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A Theory of Fuzzy Uniformities with Applications to the Fuzzy Real Lines

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For each completely distributive lattice L with order-reversing involution, the fuzzy real line $\mathbb{R}(L)$ is uniformizable by a uniformity which both generates the canonical (fuzzy) topology and induces a pseudometric generating the canonical topology. If L is also a chain, the usual addition and multiplication defined on $\mathbb{R} \equiv \mathbb{R}(\{0, 1\})$ extend jointly (fuzzy) continuously to $\oplus$ and $\odot$ on $\mathbb{R}(L)$. Three fundamental questions in fuzzy sets until now are:

Question A. If $L_1 \simeq L_2$, is $\mathbb{R}(L_1)$ uniformly isomorphic to $\mathbb{R}(L_2)$ in some sense?

Question B. For each chain L, is $\oplus$ (jointly) uniformly continuous in a sense which guarantees its (joint) continuity on $\mathbb{R}(L)$?

Question C. Is $\mathbb{R}(L)$ a complete pseudometric space in some sense?

We construct categories $\mathbb{QU}$ and $\mathbb{U}$ using the [quasi-] uniformities of B. Hutton which enable us to answer these questions in the affirmative. These results enhance the canonical standing of the fuzzy real lines and so give additional justification for answering in the affirmative:

Question D. Does fuzzy topology have deep, specific, canonical examples?

1. Introduction

Throughout this paper L is a complete lattice with bounds 0 and 1. Unless stated otherwise, only when speaking of quasi-uniform spaces (defined in Sect. 2) is L assumed to be completely distributive, and only when speaking of uniform spaces and the fuzzy real lines (defined below) is L assumed to be completely distributive and to have an order reversing involution $\alpha \to \alpha'$ (and so is a DeMorgan algebra). An important example of L in fuzzy set theory is $I = [0, 1]$ with $\alpha' = 1 - \alpha$.

Let X be a set and L a lattice. Elements of L^X are (L-) *fuzzy sets* in X [75, 12]. If $\tau \subset L^X$ is closed under (arbitrary) suprema and finitely indexed infima (and so by convention contains the constant maps 0, 1), τ is

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an (L-) *fuzzy topology* on X and (X, L, τ) is an (L-) *fuzzy topological space* [13, 21]. If $|L| = 2$, then $L^X \simeq \mathscr{P}(X)$ and τ is an ordinary topology on X. Since $|L| > 2$ is always allowed in this paper, the adjective "fuzzy" is to be understood when speaking of topologies, continuity, etc., and so is often dropped.

The (*L-fuzzy*) *real line* $\mathbb{R}(L)$ [21, 11] is the set of all equivalence classes $[\lambda]$, where $\lambda: \mathbb{R} \to L$ is monotone decreasing, $\lambda((+\infty)-) = 0$, $\lambda((-\infty)+) = 1$, and $\mu \in [\lambda]$ iff $\lambda(t-) = \mu(t-)$ for each $t \in \mathbb{R}$. We generally write λ for both $[\lambda]$ and the representative λ; this abuse is inconsequential [41, 42]. The *canonical* (*L-fuzzy*) *topology* on $\mathbb{R}(L)$ has subbasis $\{L_t, R_t : t \in [-\infty, +\infty]\}$, where for $t \in \mathbb{R}$, $L_t(\lambda) = (\lambda(t-))'$, $R_t(\lambda) = \lambda(t+)$; for $t = +\infty$, $L_t \equiv 1$ and $R_t \equiv 0$, and for $t = -\infty$, $L_t \equiv 0$ and $R_t \equiv 1$. By $\mathbb{R}(L)$ we also intend the set $\mathbb{R}(L)$ with the canonical topology. The real lines are essentially fuzzy-topologized spaces of distribution functions (cf. [17, 30, 41]).

The real lines are important, canonical examples of (fuzzy) topological spaces. For each L, $\mathbb{R}\{0, 1\} \subset \mathbb{R}(L)$ and $\mathbb{R}\{0, 1\}$ is topologically isomorphic to $\mathbb{R}$ [21, 11]. The partial ordering on $\mathbb{R} \simeq \mathbb{R}\{0, 1\}$ extends to $\mathbb{R}(L)$, and for L a chain, addition and multiplication extend jointly (fuzzy) continuously to $\oplus$ [60] and $\odot$ [64] on $\mathbb{R}(L)$: for each chain L, $\mathbb{R}(L)$ is a complete fuzzy topological hyperfield [60, 64]. Many topological properties are possessed by $\mathbb{R}(L)$: $\mathbb{R}(L)$ and the (L-fuzzy) unit interval $I(L)$ [21] satisfy many separation axioms [54, 61, 63, 43]—e.g., for each L the canonical topology of $\mathbb{R}(L)$, $I(L)$ is induced by the canonical uniformity in the sense of [22, 8, 63] which also induces a pseudometric inducing the canonical topology [63]; $I(L)$ exhibits various compactness conditions under various lattice conditions [11, 32, 61]; $\mathbb{R}(L)$, $I(L)$ exhibit various connectedness conditions under various lattice conditions [59, 61]; and the uniform, normal, and perfectly normal spaces as defined in [21, 22] are characterized using $I(L)$ by the Hutton–Ürysohn lemmas [21, 22, 62]. Studies of these real lines, other fuzzy real lines, and other canonical examples of (fuzzy) topological spaces include [2–6, 11, 14–22, 27–29, 32, 40–44, 54, 56–64].

In [62] we constructed the category $\mathbb{T}$ of fuzzy topological spaces (called FUZZ in [62] and defined in Sect. 2): it is the smallest category in the literature containing all the real lines viewed as (fuzzy) topological spaces. But there has been no categorical framework for all the real lines viewed as (fuzzy) uniform spaces in the sense of [22, 8, 63]. Thus, a fundamental question of fuzzy sets is

Question A. Is it possible to compare two real lines as uniform spaces; in particular, is $\mathbb{R}(L_1)$ uniformly isomorphic to $\mathbb{R}(L_2)$ in some sense if $L_1 \simeq L_2$?

In [38], Lowen constructed a category of (fuzzy) uniform spaces for the lattice $L=I$ (with $\alpha'=1-\alpha$). He then showed in [41] how to use hyperspaces to place a uniformity of [38] on $\mathbb{R}(I)$ which made $\oplus$ of [60] uniformly continuous. But this uniformity is not compatible with the canonical topology on $\mathbb{R}(I)$ and so this uniform continuity of $\oplus$ does not imply the continuity of $\oplus$ on $\mathbb{R}(I)$ (nor the continuity of $+$ on $\mathbb{R}\{0, 1\}$). Thus, a second fundamental question of fuzzy sets is

Question B. For each chain L, does $\oplus$ possess a type of uniform continuity which guarantees its continuity on $\mathbb{R}(L)$? (See Question 7.2 of [64].)

Completeness and completions of the uniform spaces of [19 and 38] have been studied extensively by Höhle [19] and Lowen and Wuyts [46, 47]. A fundamental question which remains is

Question C. In what sense are the real lines complete either as uniform spaces or as pseudometric spaces?

It is the primary purpose of this paper to answer Questions A, B, and C in the affirmative. More precisely, we obtain the following results:

(1) Using the quasi-uniform spaces of Hutton [22] we construct in Section 3 a new category $\mathbb{QU}$ of quasi-uniform spaces and morphisms such that the functor mapping $\mathbb{QU}$ into $\mathbb{T}$ is a natural extension of that mapping UNIF into TOP.

(2) In Section 4 we construct a new theory of neighborhoods of fuzzy sets which characterizes the topology generated by a quasi-uniformity.

(3) We construct in Section 5 induced subspace and product quasi-uniformities and show, using our theory of neighborhoods, that the subspace and product quasi-uniformities induce the usual subspace and product (fuzzy) topologies.

(4) In Section 6 we answer Question A in a full subcategory of $\mathbb{QU}$, i.e., the uniform spaces of Hutton [22].

(5) We answer Question B in Section 7 by showing that for each L, $\oplus$ is quasi-uniformly continuous with respect to the product quasi-uniformity on $\mathbb{R}(L)\times\mathbb{R}(L)$ induced from the canonical uniformity on $\mathbb{R}(L)$; this quasi-uniform continuity implies the uniform continuity of $\oplus$ with respect to the product uniformity induced on $\mathbb{R}(L)\times\mathbb{R}(L)$, the continuity of $\oplus$ on $\mathbb{R}(L)\times\mathbb{R}(L)$ equipped with the canonical product topology, and the continuity of $\oplus$ on $\mathbb{R}(I)\times\mathbb{R}(I)$ equipped with the canonical star-product topology.

(6) We answer Question C in Section 8 by showing that if $\alpha>\gamma>\alpha'$ in L^c, $\mathbb{R}(L)$ is strongly α-complete in its pseudometric.

These results are further indication of the canonical stature of the real lines and thus are additional evidence for answering in the affirmative the following question (see [40, 41] for a closed related philosophical question):

Question D. Does fuzzy topology have deep, specific, canonical examples?

Another philosophical question may be addressed by these results. Let (X, L, τ) be a topological space, let σ be the collection of all constant maps in L^X, and put $\tau^c = \tau \vee \sigma$; (X, L, τ^c) is a *stratified fuzzy topological space* in accordance with [52, 53] and $\mathbb{T}_k$ [62] (called $\mathscr{C}_k$ in [62]) is the full subcategory of $\mathbb{T}$ of such spaces. Let $\mathbb{R}^c(L)$ [62] denote the stratification of $\mathbb{R}(L)$. Question B is a valid question if "$\mathbb{R}(L)$" is replaced by "$\mathbb{R}^c(L)$", but the solution is an immediate consequence of (5) above (Section 7). Thus, the simplest solution for this question posed about objects in $\mathbb{T}_k$ uses objects in $\mathbb{T} - \mathbb{T}_k$. This, in addition to the arguments of [62] and Section 9, would seem to answer in the negative

Question E. Is $\mathbb{T}$ too general a framework for fuzzy topology?

An alternate viewpoint is suggested in [33–41, 44, 45], where the notion of including all constant maps in each topology was first proposed [33] and developed [33–41, 44, 45]. See also the bibliographies of [33–41, 44, 45] and also [10, 14–20].

Some comparisons of our approach with the uniformities of the usual set theory, the uniformities of Lowen [38], and the T-uniformities of Höhle [18] are deferred to Section 9; in this section we also comment on the generality of the categories $\mathbb{QU}$ and $\mathbb{U}$ (and the implications for Questions D and E) and state several open questions prompted by our results and methods. We give the needed preliminaries in Section 2.

2. Preliminary Notions

Definition 2.1 [62]. By $\mathbb{T}$ (called FUZZ in [62]) we intend that category of objects and morphisms as follows:

(1) The objects are fuzzy topological spaces (as defined in Sect. 1).

(2) Let $f: X_1 \to X_2$ be a function, let $\phi^{-1}: L_2 \to L_1$ be a lattice morphism (preserving (arbitrary) $\vee$ and $\wedge$, and also $'$ if L_1, L_2 have $'$; ϕ is only assumed to be a relation from range of ϕ^{-1} to L_2), let $f^{-1}: L_1^{X_2} \to L_1^{X_1}$ by $f^{-1}(b) = b \circ f$, and let $F_\phi^{-1}: L_2^{X_2} \to L_1^{X_1}$ by $F_\phi^{-1} = \phi^{-1} \circ f^{-1}(b)$. Then $(f, \phi): (X_1, L_1, \tau_1) \to (X_2, L_2, \tau_2)$ is a morphism if $F_\phi^{-1}(v) \in \tau_1$ for each $v \in \tau_2$ (cf. [23]).

(3) $(f_1, \phi_1) \circ (f_2, \phi_2) = (f_1 \circ f_2, \phi_1 \circ \phi_2)$.

The justification for calling the lattice morphism ϕ^{-1} instead of ϕ can be seen in [23]. The morphism (f, ϕ) is *(fuzzy) continuous*; it is also a *(fuzzy) homeomorphism* if f is a bijection, ϕ is an isomorphism, and (f^{-1}, ϕ^{-1}) is also a morphism. If $\phi = i_L$ (the identity on L), (f, i_L) is also represented by f (since $F_{\phi}^{-1} = f^{-1}$), in which case f is (fuzzy) continuous. We also need $f_{\text{ind}}: L_1^{X_1} \to L_1^{X_2}$ defined by $f_{\text{ind}}(a)(y) = \bigvee \{a(x): f(x) = y\}$; we will write $f(a)$ for $f_{\text{ind}}(a)$.

Let $(X, L, \tau) \in |\mathbb{T}|$ and $A \subset X$. By $\mu(A)$ we mean the characteristic function for A defined from X into L. The *subspace (fuzzy) topology* on A is $\tau(A) = \{u|_A : u \in \tau\}$ [71] (fuzzy subspaces based on *fuzzy* subsets are studied in [9, 10, 63]). The formal definitions of *basis* and *subbasis* are given in [11, 71].

Let $\mathbb{T}_k$ be that full subcategory of $\mathbb{T}$ in which for each object (X, L, τ), τ contains all the constant maps σ in L^X. For $(X, L, \tau) \in |\mathbb{T}|$, put $\tau^c = \tau \vee \sigma$, and define $G_k: \mathbb{T} \to \mathbb{T}_k$ by

$$G_k(X, L, \tau) = (X, L, \tau^c)$$

$$G_k(f, \phi) = (f, \phi).$$

PROPOSITION 2.1 [64]. *G_k is a faithful functor.*

Let $\mathbb{T}(L, \phi)$ be that subcollection of objects and morphisms of $\mathbb{T}$ in which L and ϕ are fixed; $\mathbb{T}_k(L, \phi)$ is analogously defined. Note $\mathrm{TOP} \simeq \mathbb{T}_{(\{0,1\}, i_{\{0,1\}})}$. The approach to fuzzy topology in [10, 12, 20, 21, 26–28, 31, 45, 51–60, 68–73] is represented by $\mathbb{T}(L, i_L)$, that approach in [10, 33–41, 44, 45] by $\mathbb{T}_k(I, i_I)$ (note in these papers $|\mathbb{T}_k(I, i_I)|$ are called fuzzy topological spaces and $|\mathbb{T}(I, i_I)| - |\mathbb{T}_k(I, i_I)|$ are called *quasi*-fuzzy topological spaces), and that approach in [23, 24] by the full subcategory of $\mathbb{T}$ of singleton spaces ($|X| = 1$). Heuristically, $\mathbb{T}$ is the smallest category which generalizes TOP, accomodates these approaches, and contains all real lines; it also generates in a natural way the framework for topological spaces in which the open sets exhibit second-order fuzziness (see [62] and cf. Question E of Section 1).

DEFINITION 2.2. Let $\{(X_\gamma, L, \tau_\gamma)\}_\gamma \subset |\mathbb{T}(L, \phi)|$ and let $\pi_\beta: \times_\gamma X_\gamma \to X_\beta$ denote the projection.

(1) The *ϕ-product topology* $\times_\phi \tau_\gamma$ has subbasis $\{\phi^{-1} \circ \pi_\gamma^{-1}(v): v \in \tau_\gamma\}_\gamma$ [64]. If $\phi = i_L$, the ϕ-product topology becomes the categorical Goguen–Wong product on $T(L, i_L)$ [13, 74] and is denoted by $\times \tau_\gamma$. The *canonical product topology* on $\mathbb{R}(L) \times \mathbb{R}(L)$ is the Goguen–Wong product induced from the canonical topology on $\mathbb{R}(L)$.

(2) Let $L = I$ with $\alpha' = 1 - \alpha$. The *star product topology* $*$ has subbasis

$$\{u_{\gamma_1} * \cdots * u_{\gamma_n} * \mu(\times_{\gamma \neq \gamma_i} X_\gamma): u_{\gamma_i} \in \tau_{\gamma_i} \text{ for each } i\},$$

where $(a * b)(x, y) = a(x) \cdot b(y)$ [11, 63]. The *canonical star product topology* on $\mathbb{R}(I) \times \mathbb{R}(I)$ is that induced from the canonical topology on $\mathbb{R}(I)$.

PROPOSITION 2.2 [63]. *For $L = I$, the Goguen–Wong product is contained in the star product.*

DEFINITION 2.3 [22]. Let X be a set and L a lattice. By a *quasi-uniformity* on X, we mean a set of maps $\mathscr{U} \subset (L^X)^{(L^X)}$ which satsify:

(1) $\mathscr{U} \neq \phi$

(2) $\forall a \in L^X, \forall U \in \mathscr{U}, a \leqslant U(a)$

(3) $\forall \{a_\gamma\}_\gamma \subset L^X, \forall U \in \mathscr{U}, U(\bigvee_\gamma a_\gamma) = \bigvee_\gamma U(a_\gamma)$

(4) $\forall V \varepsilon (L^X)^{(L^X)}$ satsifying (2, 3), $U \in \mathscr{U}$ and $U \leqslant V \Rightarrow V \in \mathscr{U}$

(5) $U_1, U_2 \in \mathscr{U} \Rightarrow \exists U \in \mathscr{U}, \quad U \leqslant U_1 \wedge U_2$ (where $(U_1 \wedge U_2)(a) = U_1(a) \wedge U_2(a)$)

(6) $U \in \mathscr{U} \Rightarrow \exists V \in \mathscr{U}, \; V \circ V \leqslant U$ (where $\circ$ denotes the usual composition).

For $U \in (L^X)^{(L^X)}$, define $U^{-1} \in (L^X)^{(L^X)}$ by

$$U^{-1}(a) = \bigwedge \{b: U(b') \leqslant a'\}.$$

Then $\mathscr{U}$ is a *uniformity* if it satisfies (1)–(6) and also

(7) $U \in \mathscr{U} \Rightarrow U^{-1} \in \mathscr{U}$

LEMMA 2.3. *For each $a \in L^X$, $\exists \mathscr{C}(a) \subset L^X$,*

(1) $\bigvee \mathscr{C}(a) = a$

(2) $A \subset L^X$ *and* $\bigvee A = a \Rightarrow \forall b \in \mathscr{C}(a), \exists c \in \mathscr{A}, b \leqslant c.$

Lemma 2.3 is originally due to Raney; we have stated it as in [22].

DEFINITION 2.4. Let $U, V \in (L^X)^{(L^X)}$. Define $U \Delta V \in (L^X)^{(L^X)}$ for U, V satisfying Definition 2.3 (2, 3) by

$$(U \Delta V)(a) = \bigvee \{(U \wedge V)(b): b \in \mathscr{C}(a)\}$$

(where $\mathscr{C}(a)$ comes from Lemma 2.3).

If U, V satisfy Definition 2.3(2), (3), then $U \Delta V$ is the largest member of $(L^X)^{(L^X)}$ which satisfies Defintion 2.3(2), (3) and is less than or equal to $U \wedge V$; also $(U \Delta U^{-1})^{-1} = U \Delta U^{-1}$ [22].

If $S \subset (L^X)^{(L^X)}$ satisfies Definition 2.3(2, 3, 6), $\mathscr{S}$ is a *subbasis* for a quasi-uniformity $\langle\!\langle \mathscr{S} \rangle\!\rangle$ on X: V is this quasi-uniformity if V satisfies (3) and $\exists S_1, ..., S_n \in \mathscr{S}$, $\Delta_i S_i \leqslant V$. If $\mathscr{S}$ additionally satisfies (5), $\mathscr{S}$ is a *basis* for this quasi-uniformity written now as $\langle \mathscr{S} \rangle$. This quasi-uniformity is a uniformity if $\mathscr{S}$ satisfies (2), (3), (5)–(7) [22].

DEFINITION 2.5 [22]. Let $(X, L, \mathscr{U})$ be a quasi-uniform space. The topology induced or generated on X by $\mathscr{U}$ is

$$\tau(\mathscr{U}) = \left\{ u \in L^X : u = \bigvee \{a \in L^X : \exists U \in \mathscr{U},\ U(a) \leqslant u\} \right\}.$$

It follows [22] that

$$\mathrm{Int}(v) = \bigvee \{a \in L^X : U \in \mathscr{U},\ U(a) \leqslant v\}$$

is the associated interior operator.

3. THE CATEGORIES $\mathbb{QU}$ AND $\mathbb{U}$

In this and later sections, "quasi-uniform" and "uniform" are understood in the sense of Definition 2.3.

DEFINITION 3.1. Construction of $\mathbb{QU}$ and $\mathbb{U}$.

I. Objects. Objects are of the form $(X, L, \mathscr{U})$, where in the case of $\mathbb{QU}$, $(X, L, \mathscr{U})$ is a quasi-uniform space (based on L), and in the case of $\mathbb{U}$, $(X, L, \mathscr{U})$ is a uniform space (based on L).

II. Morphisms. A morphism from $(X_1, L_1, \mathscr{U}_1)$ to $(X_2, L_2, \mathscr{U}_2)$ is of the form (f, ϕ_1, ϕ_2) such that the following conditions are satisfied:

(1) $f: X_1 \to X_2$ is a function.

(2) $\phi_1 : L_1 \to L_2$ and $\phi_2^{-1} : L_2 \to L_1$ are lattice morphisms (by Definition 2.1(2)); we only assume ϕ_2 is a relation. We require ϕ_1 to be a surjection, $\phi_1 \circ \phi_2^{-1} \leqslant i_{L_2}$, and $\phi_2^{-1} \circ \phi_1 \geqslant i_{L_1}$. Categorically (viewing L_1 and L_2 as pre-ordered categories), these inequalities say $\phi_1 \dashv \phi_2^{-1}$.

(3) Put $\Phi: L_1^{X_2} \to L_2^{X_2}$ by $\Phi(a) = \phi_1 \circ a$; we also designate Φ by $[\phi_1]$. Put $\Phi_f^{-1}: L_2^{X_2} \to L_1^{X_1}$ by $\Phi_f^{-1}(b) = \phi_2^{-1} \circ b \circ f$; we also designate Φ_f^{-1} by $[\phi_2]_f^{-1}$. Put $F^{-1}: (L_2^{X_2})^{(L_2^{X_2})} \to (L_1^{X_1})^{(L_1^{X_1})}$ by $F^{-1}(B) = \Phi_f^{-1} \circ B \circ \Phi \circ f_{\mathrm{ind}}$; we call F^{-1} the auxiliary *map* of (f, ϕ_1, ϕ_2). It is required that $F^{-1}(V) \in \mathscr{U}_1$ for each $V \in \mathscr{U}_2$. See Fig. 3.1.

$$
\begin{tikzcd}
L_1^{X_1} \arrow[rr, "f_{\mathrm{ind}}"] \arrow[dr, dashed, "F^{-1}(V)"'] & & L_1^{X_2} \arrow[rr, "\Phi"] \arrow[d, phantom, "\circ"] & & L_2^{X_2} \arrow[dl, "V"] \\
& L_1^{X_1} & {} & L_2^{X_2} \arrow[ll, "\Phi_f^{-1}"] &
\end{tikzcd}
$$

FIGURE 3.1

III. Composition of morphisms. $(g, \psi_1, \psi_2) \circ (f, \phi_1, \phi_2) = (g \circ f, \psi_1 \circ \phi_1, \psi_2 \circ \phi_2)$. The auxiliary map is designated $(GF)^{-1}$.

Remark 3.1. The requirements in Definition 3.1 (II(2)) that ϕ_1 be a surjection, that $\phi_1 \circ \phi_2^{-1} \leqslant i_{L_2}$, and that $\phi_2^{-1} \circ \phi_1 \geqslant i_{L_1}$ are not too restrictive. (Let L be a lattice with $|L| \geqslant 2$, $L_2 = L$, $L_1 = L \times L$ (direct product), $\phi_2^{-1}(\alpha) = (\alpha, 1)$, and $\phi_1 = \pi_1$ (projection). Then both ϕ_2^{-1} and ϕ_1 are lattice morphisms, $\phi_1 \circ \phi_2^{-1} = i_{L_1}$ and $\phi_2^{-1} \circ \phi_1 \geqslant i_{L_1}$, where ϕ_2^{-1} is an injection and ϕ_1 is a surjection, but neither is an isomorphism.) On the other hand, each is essential: Proposition 3.2 below requires ϕ_1 be a surjection, and in Section 5, Proposition 5.3 requires $\phi_1 \circ \phi_2^{-1} \leqslant i_{L_2}$ and Lemma 5.2 requires $\phi_2^{-1} \circ \phi_1 \geqslant i_{L_1}$. Also note the Adjoint Functor Theorem guarantees a unique ϕ_2^{-1} for a given ϕ_1 (and no more than one ϕ_1 for a given ϕ_2^{-1}); nonetheless, it is convenient to explicitly designate and "track" each map.

DEFINITION 3.2. A morphism (f, ϕ_1, ϕ_2) is *quasi-uniformly continuous.* It is a *quasi-uniform isomorphism* if f, ϕ_1, and ϕ_2 are bijections and both (f, ϕ_1, ϕ_2) and $(f^{-1}, \phi_1^{-1}, \phi_2^{-1})$ are morphisms. Note if (f, ϕ_1, ϕ_2) is a quasi-uniform isomorphism, the conditions $\phi_1 \circ \phi_2^{-1} \leqslant i_{L_2}$ and $\phi_2^{-1} \circ \phi_1 \geqslant i_{L_1}$ imply $\phi_1 = \phi_2$.

PROPOSITION 3.1. *$\mathbb{QU}$ is a category; $\mathbb{U}$ is a full subcategory of $\mathbb{QU}$.*

Proof. The details are straightforward; we discuss two of them. The two-sided identity for $(X, L, \mathscr{U})$ is (i_X, i_L, i_L). The composition of morphisms is a morphism if $F^{-1}(G^{-1}(B)) = (GF)^{-1}(B)$ for $B \in (L_3^{X_3})^{(L_3^{X_3})}$, i.e., if

$$\Psi \circ g_{\mathrm{ind}} \circ \Phi \circ f_{\mathrm{ind}} = [\psi_1 \circ \phi_1] \circ (g \circ f)_{\mathrm{ind}}$$

and

$$\Phi_f^{-1} \circ \Psi_g^{-1} = [\psi_2 \circ \phi_2]_{g \circ f}^{-1}.$$

We prove only the first identity. Let $a \in L_1^{X_1}$ and $z \in X_3$. Then

$$\begin{aligned}
&[\psi_1 \circ \phi_1]((g \circ f)_{\text{ind}}(a))(z) \\
&= \psi_1(\phi_1(g(f(a))(z))) \\
&= \psi_1\left(\phi_1\left(\bigvee \{a(x)\colon g(f(x)) = z\}\right)\right) \\
&= \bigvee \{\psi_1(\phi_1(a(x)))\colon g(f(x)) = z\} \\
&= \bigvee \left\{\bigvee \{(\psi_1(\phi_1(a(x))))\colon f(x) = y\}\colon g(y) = z\right\} \\
&= \bigvee \left\{\psi_1\left(\phi_1\left(\bigvee \{a(x)\colon f(x) = y\}\right)\right)\colon g(y) = z\right\} \\
&= \bigvee \{\psi_1(\{\phi_1(f(a)(y))\colon g(y) = z\}) \\
&= \psi_1\left(\bigvee \{(\phi_1 \circ f(a))(y)\colon g(y) = z\}\right) \\
&= \psi_1(g(\phi_1 \circ f(a))(z)) \\
&= (\psi_1 \circ g(\phi_1 \circ f(a)))(z) \\
&= (\Psi \circ g_{\text{ind}} \circ \Phi \circ f_{\text{ind}})(a)(z). \quad \blacksquare
\end{aligned}$$

PROPOSITION 3.2. *Let $(X, L, \mathscr{U})$ be an object in $\mathbb{QU}$, let (f, ϕ_1, ϕ_2) be a morphism in $\mathbb{QU}$, and put*

$$J_0(X, L, \mathscr{U}) = (X, L, \tau(\mathscr{U}))$$
$$J_0(f, \phi_1, \phi_2) = (f, \phi_2).$$

Then J_0 is a functor of $\mathbb{QU}$ into $\mathbb{T}$. In particular, quasi-uniform continuity implies continuity.

Proof. The only detail that merits checking is the following matter: if $(f, \phi_1, \phi_2)\colon (X_1, L_1, \mathscr{U}) \to (X_2, L_2, \mathscr{V})$ is a morphism in $\mathbb{QU}$, is $(f, \phi_2)\colon (X_1, L_1, \tau(\mathscr{U})) \to (X_2, L_2, \tau(\mathscr{V}))$ a morphism in $\mathbb{T}$? It is initially convenient to require f to be a surjection; this restriction will be removed later. We need the following lemma.

LEMMA 3.3. *If f is a surjection, then $\Phi \circ f_{\text{ind}}\colon L_1^{X_1} \to L_2^{X_2}$ is a surjection.*

Proof. For $b \in L_2^{X_2}$, define $c \in L_1^{X_2}$ by $c(y) = \bigvee \{\alpha\colon \phi_1(\alpha) = b(y)\}$ and define $a \in L_1^{X_1}$ by $a = c \circ f$. Then $\Phi(f(a)) = b$. $\blacksquare$

Let $v \in \tau(\mathscr{V})$. Then

$$v = \bigvee \{b\colon V(b) \leqslant v \text{ for some } V \in \mathscr{V}\}$$
$$= \bigvee \{V(b)\colon b \in L_2^{X_2},\ V \in \mathscr{V}_b\},$$

where $\mathscr{V}_b = \{V \in \mathscr{V}\colon V(b) \leqslant v\}$.

Computation yields

$$F_{\phi_2}^{-1}(v) = \phi_2^{-1} \circ \left[\bigvee \{V(b)\colon b \in L_2^{X_2},\ V \in \mathscr{V}_b\}\right] \circ f$$
$$= \bigvee \{\phi_2^{-1} \circ V(b) \circ f\colon b \in L_2^{X_2},\ V \in \mathscr{V}_b\}$$
$$= \bigvee \{\Phi_f^{-1}(V(b))\colon b \in L_2^{X_2}, V \in \mathscr{V}_b\}$$
$$= \bigvee \{(\Phi_f \circ V)(b)\colon b \in L_2^{X_2},\ V \in \mathscr{V}_b\}$$
$$= \bigvee \{\Phi_f^{-1}(V(\Phi(f(a))))\colon a \in L_1^{X_1},\ V \in \mathscr{V}_{\Phi(f(a))}\}$$
$$= \bigvee \{F^{-1}(V)(a)\colon a \in L_1^{X_1},\ V \in \mathscr{V}_{\Phi(f(a))}\}, \tag{3.1}$$

where (3.1) is assured by Lemma 3.3. Now put

$$F^{-1}(\mathscr{V})_a = \{F^{-1}(V)\colon V \in \mathscr{V}, F^{-1}(V)(a) \leqslant \Phi_f^{-1}(v)\}$$
$$\mathscr{U}_a = \{U \in \mathscr{U}\colon U(a) \leqslant \Phi_f^{-1}(v)\}.$$

Then

$$F_{\phi_2}^{-1}(v) \leqslant \bigvee \{F^{-1}(V)(a)\colon a \in L_1^{X_1}, F^{-1}(V) \in F^{-1}(\mathscr{V})_a\}$$
$$\leqslant \bigvee \{U(a)\colon a \in L_1^{X_1},\ U \in \mathscr{U}_a\}$$
$$\leqslant \Phi_f^{-1}(v) = F_{\phi_2}^{-1}(v), \tag{3.2}$$

where (3.2) is assured by the quasi-uniform continuity of (f, ϕ_1, ϕ_2). But there is equality at (3.2); so $F_{\phi_2}^{-1}(v) \in \tau(\mathscr{U})$ and (f, ϕ_2) is continuous.

To remove the restriction that f be a surjection, we need the following discussion.

DISCUSSION 3.4. Let $\mathscr{U}$ be a quasi-uniformity on X and let $A \subset X$. For each $a \in L^A$, put $a^e = a$ on A and $a^e = 0$ on $X - A$. For each $a \in L^A$ and $U \in \mathscr{U}$, put $U_A(a) = U(a^e)\,|\,A$ and put $\mathscr{U}_A = \{U_A\colon U \in \mathscr{U}\}$.

(1) $\mathscr{U}_A$ is a quasi-uniformity on A; we call $\mathscr{U}_A$ the *subspace quasi-uniformity* on A. Requirements (1)–(3) and (5) of Definition 2.3 are clear. For (4), let $U_A \leqslant V$, where $V \in (L^A)^{(L^A)}$. Define $V^e \in (L^X)^{(L^X)}$ by $V^e(b) = (V(b \mid A))^e \vee U(b)$, where U_A is induced from U. It is straightforward to check that V^e satisfies (2), (3) and that $V^e \geqslant U$. Hence $V^e \in \mathscr{U}$. Computation verifies $V = V^e_A$. For (6), let U induce U_A; there is $V \in \mathscr{U}$, $V \circ V \leqslant U$. Then computation shows $V_A \circ V_A \leqslant (V \circ V)_A \leqslant U_A$.

(2) Let the subspace topology on A in $(X, \mathscr{U})$ be denoted $\tau_A(\mathscr{U})$. Then $\tau(\mathscr{U}_A) = \tau_A(\mathscr{U})$. To see this, let $u \in \tau(\mathscr{U})$, set $v = u \mid A$, and set

$$\tilde{v} = \bigvee \{b \in L^A \colon U_A(b) \leqslant u \mid A, \text{ some } U_A \in \mathscr{U}_A\}.$$

We have then

$$\begin{aligned} v &= \bigvee \{a \mid A \colon a \in L^X, U(a) \leqslant u, \text{ some } U \in \mathscr{U}\} \\ &\leqslant \bigvee \{b \in L^A \colon U(b^e) \mid A \leqslant u \mid A \text{ some } U \in \mathscr{U}\} \qquad (3.3) \\ &\leqslant \bigvee \{U(b^e) \mid A \colon b \in L^A, U(b^e) \mid A \leqslant u \mid A\} \\ &\leqslant u \mid A \\ &= v. \end{aligned}$$

This forces equality at (3.3), i.e., $v = \tilde{v}$. But $\tilde{v} \in \tau(\mathscr{U}_A)$ since $\tilde{v} = \operatorname{Int}(v)$ in $\tau(\mathscr{U}_A)$. Thus $\tau(\mathscr{U}_A) \supset \tau_A(\mathscr{U})$. The same sequence of steps, appropriately relabeled, establishes $\tau(\mathscr{U}_A) \subset \tau_A(\mathscr{U})$.

To complete the proof of Proposition 3.2, let $(f, \phi_1, \phi_2) \colon (X_1, L_1, \mathscr{U}) \to (X_2, L_2, \mathscr{V})$ be a morphism. It follows that (f, ϕ_1, ϕ_2) is a morphism into $(f(X_1), L_2, \mathscr{V}_{f(X_1)})$—this is a straightforward consequence of Discussion 3.4(1) and the following fact: if $a \in L_1^{X_2}$, then $a \mid f(X_1) \circ f = a \circ f$. By the surjective case above, $(f, \phi_2) \colon (X_1, L_1, \tau(\mathscr{U})) \to (f(X_2), L_2, \tau(\mathscr{V}_{f(X_1)})$ is a morphism. From Discussion 3.4(2), we have (f, ϕ_2) is a morphism into $(f(X_2), L_2, \tau_{f(X_1)}(\mathscr{V}))$. It now follows that (f, ϕ_2) is a morphism into $(X_2, L_2, \tau(\mathscr{V}))$ by the fact cited third sentence above. ∎

Remark 3.2. If $\mathscr{U}$ is a uniformity on X, then $\mathscr{U}_A$ may only be a quasi-uniformity on $A \subset X$. But $\mathscr{U}_A$ induces a uniformity on A. This is discussed formally in Section 5. Also note that quasi-uniformities on a *fuzzy* subset are discussed in [63].

PROPOSITION 3.5. *Let* $\mathbb{QU}(L, \phi_1, \phi_2)$ *denote that subcollection of objects and morphisms of* $\mathbb{QU}$ *in which all objects have the same underlying lattice L* *and all morphisms have the same second and third components* ϕ_1, ϕ_2, *and let* $\mathbb{U}(L, \phi_1, \phi_2)$ *be analogously defined*:

(1) *The Hutton approach to quasi-uniformities [uniformities] may be identified with a class of unrelated categories, each of the form* $\mathbb{QU}(L, i_L, i_L)$ $[\mathbb{U}(L, i_L, i_L)]$.

(2) *The category* UNIF [TOP] *may be identified with* $\mathbb{U}(\{0, 1\}, i_{\{0,1\}}, i_{\{0,1\}})$ $[\mathbb{T}(\{0, 1\}, i_{\{0,1\}})]$.

(3) *Let* J_1 *be that functor mapping a particular* $\mathbb{QU}(L, i_L, i_L)$ *into* $\mathbb{T}(L, i_L)$ *given by Proposition* 8 *of* [22], *and let* J_2 *be the usual functor mapping* UNIF *into* TOP. *Then*

$$J_1 = J_0 \mid \mathbb{QU}(L, i_L, i_L) \qquad \text{and} \qquad J_2 = J_0 \mid \mathrm{UNIF},$$

where J_0 *is given in Proposition* 3.2 *above.*

Remark 3.3. "Quasi-full subcategories of $\mathbb{QU}$" may no doubt be defined analogous to "quasi-full subcategories of $\mathbb{T}$" and in more than one way (cf. Definition 3.5 of [62]). Under one such definition, the quasi-full subcategories of $\mathbb{QU}$ map to quasi-full subcategories of $\mathbb{T}$ via J_0. We do not touch on the question of characterizing such subcategories, but we note that the analogous question for $\mathbb{T}$ (Question 3.1 of [62]) has been recently answered by Eklund [7] and that this solution says that all quasi-full subcategories of $\mathbb{T}_k$ are full.

4. Neighborhood Systems of Fuzzy Sets

In this section we develop a theory of neighborhoods of fuzzy set which we use to characterize fuzzy topologies, continuity, and, in particular, the topologies induced by quasi-uniformities (cf. [55]). Such results are essential for the latter results of the next section which are crucial for Section 7. At least three other theories of fuzzy neighborhoods have appeared: Lowen [39], Ludescher and Roventa [48], and Warren [72] (also see [52, 53]). The theory of [39] characterizes topologies generated by uniformities of [38], that of [48] does not characterize fuzzy topologies, and that of [72] gives a "point-dependent" characterization of fuzzy topologies. In any case, a new theory is needed to characterize topologies induced by the "point-free" quasi-uniformities of [22]. Our development partly parallels [72 and 68, Chap. 9]. We abbreviate neighborhood by "nbhd."

DEFINITION 4.1. Let (X, L, τ) be a topological space and let $a, N \in L^X$. We say N is a *nbhd of a* iff $\exists u \in \tau$, $a \leqslant u \leqslant N$. Put

$$\mathcal{N}_a(\tau) = \{N\colon N \text{ is a nhbd of } a\}$$

$$\mathcal{N}(\tau) = \bigcup \{\mathcal{N}_a(\tau)\colon a \in L^X\}.$$

PROPOSITION 4.1. *Let (X, L, τ) be a topological space. Then for each $b \in L^X$, $b \in \tau$ iff $[a \leqslant b \Rightarrow b \in \mathcal{N}_a(\tau)]$.*

PROPOSITION 4.2. *Let (X, L, τ) be given. The following statements hold:*

P1. $0 \in \mathcal{N}_0$ *and* $\forall a \in L^X$, $1 \in \mathcal{N}_a(\tau)$ *(0 is the zero constant map).*

P2. $a \leqslant b \Rightarrow \mathcal{N}_b(\tau) \subset \mathcal{N}_a(\tau)$.

P3. $\forall N \in \mathcal{N}_a(\tau)$, $a \leqslant N$.

P4. $N \in \mathcal{N}_a(\tau)$, $N \leqslant W \Rightarrow W \in \mathcal{N}_a(\tau)$.

P5. $N_\alpha \in \mathcal{N}_{a_\alpha}(\tau)$ *for each* $\alpha \Rightarrow \bigvee_\alpha N_\alpha \in \mathcal{N}_{\bigvee_\alpha a_\alpha}(\tau)$ *(α is merely an index).*

P6. $N, M \in \mathcal{N}_a(\tau) \Rightarrow N \wedge M \in \mathcal{N}_a(\tau)$.

P7. $N \in \mathcal{N}_a(\tau) \Rightarrow \exists M \in \mathcal{N}_a(\tau)$, $M \leqslant N$, *and* $b \leqslant M \Rightarrow M \in \mathcal{N}_b(\tau)$.

Proof. Each is clear; e.g., P5 follows since $a_\alpha \leqslant u_\alpha \leqslant N_\alpha$ for each α implies $\bigvee_\alpha a_\alpha \leqslant \bigvee_\alpha u_\alpha \leqslant \bigvee_\alpha N_\alpha$. ∎

DEFINITION 4.2. We say that $\theta\colon L^X \to \{0, 1\}^{(L^X)}$ is a *(fuzzy) nbhd system on X if the* following axioms hold:

N1. $0 \in \theta(0)$ and $\forall a \in L^X$, $1 \in \theta(a)$ (0 is the zero constant map).

N2. $a \leqslant b \Rightarrow \theta(b) \subset \theta(a)$.

N3. $N \in \theta(a) \Rightarrow a \leqslant N$.

N4. $N \in \theta(a)$, $N \leqslant W \Rightarrow W \in \theta(a)$.

N5. $N_\alpha \in \theta(a_\alpha)$ for each $\alpha \Rightarrow \bigvee_\alpha N_\alpha \in (\bigvee_\alpha a_\alpha)$ (α is merely an index).

N6. $N, M \in \theta(a) \Rightarrow N \wedge M \in \theta(a)$.

N7. $N \in \theta(a) \Rightarrow \exists M \in \theta(a)$, $M \leqslant N$, and $b \leqslant M \Rightarrow M \in \theta(b)$.

If θ is a nbhd system on X, we put

$$\tau(\theta) = \{N\colon a \leqslant N \Rightarrow N \in \theta(a)\}.$$

PROPOSITION 4.3. *If θ is a nbhd system on X, then $\tau(\theta)$ is an L-fuzzy topology on X.*

Proof. Only the requirement that $\tau(\theta)$ be closed under (arbitrary) suprema merits checking. Let $\{N_\alpha\}_\alpha \subset \tau(\theta)$ and let $a \leqslant \bigvee_\alpha N_\alpha$. Fix α, fix $x \in X$, and put

$$b_{x\alpha}(y) = \begin{cases} N_\alpha(y), & y = x. \\ 0, & y \neq x. \end{cases}$$

Then $b_{x\alpha} \in L^X$ and $b_{x\alpha} \leqslant N_\alpha$. So $N_\alpha \in \theta(b_{x\alpha})$. By N5, $\bigvee_\alpha N_\alpha \in \theta(\bigvee_x \bigvee_\alpha b_{x\alpha})$. Trivially $a \leqslant \bigvee_x \bigvee_\alpha b_{x\alpha}$ on X, so by N2 we have $\bigvee_\alpha N_\alpha \in \theta(a)$. ∎

PROPOSITION 4.4. $\forall a \in L^X$, $\mathcal{N}_a(\tau(\theta)) = \theta(a)$.

Proof. One inclusion follows from the definition of $\tau(\theta)$ and N4. The reverse inclusion follows from N3 and N7. ∎

PROPOSITION 4.5. *Let* (X, L, τ) *be a topological space and put* $\theta_\tau(a) = \mathcal{N}_a(\tau)$ *for each* $a \in L^X$. *Then* $\tau(\theta_\tau) = \tau$.

Proof. Because of Propositions 4.2, 4.3, and 4.4, θ_τ satisfies N1–N7 and hence $\tau(\theta_\tau)$ is an L-fuzzy topology on X such that $\forall a \in L^X$, $\mathcal{N}_a(\tau(\theta_\tau)) = \theta_\tau(a)$, i.e., $\mathcal{N}_a(\tau(\theta_\tau)) = \mathcal{N}_a(\tau)$. Proposition 4.1 now implies the two topologies are identical. ∎

COROLLARY 4.6. *Let* τ_1, τ_2 *be L-fuzzy topologies on X. Then* $\tau_1 \subset \tau_2$ *iff* $\mathcal{N}(\tau_1) \subset \mathcal{N}(\tau_2)$.

DEFINITION 4.3. Let (X, L, τ) be a topological space. We say $\mathcal{B} \subset \mathcal{N}(\tau)$ is a *basis for* $\mathcal{N}(\tau)$ iff $\forall a \in L^X$, $\forall N \in \mathcal{N}_a(\tau)$, $\exists \{a_\alpha\}_\alpha \subset L^X$, $\exists \{B_\alpha\}_\alpha \subset B$ such that

(i) $\forall \alpha$, $B_\alpha \in \mathcal{N}_{a_\alpha}(\tau)$

(ii) $a = \bigvee_\alpha a_\alpha \leqslant \bigvee_\alpha B_\alpha \leqslant N$.

We say $\mathcal{S} \subset \mathcal{N}(\tau)$ is a *subbasis for* $\mathcal{N}(\tau)$ if the collection of all finitely indexed infima of members of $\mathcal{S}$ is a basis for $\mathcal{N}(\tau)$. For convenience, we may write $\mathcal{N}(\tau) = \langle \mathcal{B} \rangle = \langle\!\langle \mathcal{S} \rangle\!\rangle$. Trivially, $\mathcal{N}(\tau) = \{N\colon N \geqslant B$ for some $B \in \mathcal{B}\}$ by N4 and N5.

DEFINITION 4.4. Let $f\colon X_1 \to X_2$ be a function, $\phi^{-1}\colon L_2 \to L_1$ be a lattice morphism, and (X_1, L_1, τ_1) and (X_2, L_2, τ_2) be topological spaces. We say F_ϕ^{-1} *preserves nbhds* if $\forall b \in L_2^{X_2}$, $N \in \mathcal{N}_b(\tau_2)$, $F_\phi^{-1}(N)$ is a nbhd of $F_\phi^{-1}(b)$. The definition of *preserves basic* [*subbasic*] *nbhds* is similar.

PROPOSITION 4.7. *Let* $f\colon X_1 \to X_2$ *be a function,* $\phi^{-1}\colon L_2 \to L_1$ *be a lattice morphism,* (X_1, L_1, τ_1) *and* (X_2, L_2, τ_2) *be topological spaces, and* $\mathcal{N}(\tau_2)$ *have basis* $\mathcal{B}$ *induced from subbasis* $\mathcal{S}$. *The following are equivalent*:

(1) *(f, ϕ) is continuous.*

(2) *F_ϕ^{-1} preserves nbhds.*

(3) *F_ϕ^{-1} preserves the basic nbhds from $\mathscr{B}$.*

(4) *F_ϕ^{-1} preserves the subbasic nbhds from $\mathscr{S}$.*

Proof. (4) $\Rightarrow$ (3). Consequence of N6 and the fact that F_ϕ^{-1} preserves $\bigwedge$.

(3) $\Rightarrow$ (2). Consequence of N5, N4, and the fact that F_ϕ^{-1} preserves $\vee$ and $\leqslant$.

(2) $\Rightarrow$ (1). Consequence of N2 and Proposition 4.1.

(1) $\Rightarrow$ (2). Consequence of the fact that F_ϕ^{-1} preserves $\leqslant$.

The other needed implications are trivial. ∎

PROPOSITION 4.8. *Let $(X, L, \mathscr{U})$ be a quasi-uniform space with $\mathscr{U} = \langle \mathscr{B} \rangle = \langle\langle \mathscr{S} \rangle\rangle$ and put $\bar{\mathscr{U}} = \{U(a): a \in L^X,\ U \in \mathscr{U}\}$, $\bar{\mathscr{B}} = \{B(a): a \in L^X, B \in \mathscr{B}\}$, and $\bar{\mathscr{S}} = \{S(a): a \in L^X,\ S \in \mathscr{S}\}$. Then $\mathscr{N}(\tau(\mathscr{U})) = \langle \bar{\mathscr{U}} \rangle = \langle \bar{\mathscr{B}} \rangle = \langle\langle \bar{\mathscr{S}} \rangle\rangle$.*

Proof. First note $\bar{\mathscr{B}} \subset \bar{\mathscr{U}} \subset \mathscr{N}(\tau(\mathscr{U}))$ since $a \leqslant \mathrm{Int}(U(a)) \leqslant U(a)$. To show $\langle \bar{\mathscr{S}} \rangle \subset \mathscr{N}(\tau(\mathscr{U}))$, let $\bigwedge_{i=1}^n S_i(a_i) \in \langle \bar{\mathscr{S}} \rangle$ and put $a = \bigwedge_i a_i$. Then $a \leqslant a_i \leqslant S_i(a_i)$, $S_i(a_i) \in \mathscr{N}_a(\tau(\mathscr{U}))$, and $\bigwedge_i S_i(a_i) \in \mathscr{N}_a(\tau(\mathscr{U}))$.

To see that each of $\bar{\mathscr{U}}$, $\bar{\mathscr{B}}$, and $\langle \bar{\mathscr{S}} \rangle$ is a basis for $\mathscr{N}(\tau(\mathscr{U}))$, we begin with $\bar{\mathscr{U}}$. Let $N \in \mathscr{N}_a(\tau)$. There is $u \in \tau(\mathscr{U})$, $a \leqslant u \leqslant N$, where

$$u = \bigvee \{v: U(v) \leqslant u, \text{ some } U \in \mathscr{U}\}.$$

For each such v, we have

$$a \wedge v \leqslant U(a \wedge v) \leqslant U(v)$$

so that

$$\begin{aligned} a &= \bigvee \{a \wedge v: \text{such } v\} \\ &\leqslant \bigvee \{U(a \wedge v): \text{such } v, U\} \\ &\leqslant \bigvee \{U(v): \text{such } v, U\} \\ &= u \leqslant N. \end{aligned}$$

For the $\bar{\mathscr{B}}$ case, consider a U under discussion in the $\bar{\mathscr{U}}$ case. There is $B \in \mathscr{B}$, $B \leqslant U$. So

$$a \wedge v \leqslant B(a \wedge v) \leqslant U(a \wedge v)$$

and

$$a = \bigvee \{a \wedge v\colon \text{such } v\} \tag{4.1}$$
$$\leqslant \bigvee \{B(a \wedge v)\colon \text{such } v, B\}$$
$$\leqslant N.$$

The $\langle \bar{\mathscr{S}} \rangle$ case is more delicate. Consider a B under discussion in the $\bar{\mathscr{B}}$ case. Then $B = \Delta_{i=1}^{n} S_i$, where $S_i \in \mathscr{S}$. The argument (by induction) is the same as for $B = S_1 \Delta S_2 \Delta S_3$. From Definition 2.4,

$$\begin{aligned} B(a \wedge v) &= ((S_1 \Delta S_2) \Delta S_3)(a \wedge v) \\ &= \bigvee \left\{ \left(\bigvee \{(S_1 \wedge S_2)(c)\colon c \in \mathscr{C}(b)\} \right) \wedge S_3(b)\colon b \in \mathscr{C}(a \wedge v) \right\} \\ &= \bigvee \left\{ \bigvee \{S_1(c) \wedge S_2(c) \wedge S_3(b)\colon c \in \mathscr{C}(b)\}\colon b \in \mathscr{C}(a \wedge v) \right\} \\ &\geqslant \bigvee \left\{ \bigvee \{S_1(c) \wedge S_2(c) \wedge S_3(c)\colon c \in \mathscr{C}(b)\}\colon b \in \mathscr{C}(a \wedge v) \right\}. \end{aligned}$$

Now

$$a \wedge v = \bigvee \{b\colon b \in \mathscr{C}(a \wedge v)\} = \bigvee \left\{ \bigvee \{c\colon c \in \mathscr{C}(b)\}\colon b \in \mathscr{C}(a \wedge v) \right\}$$

and for each such c

$$c \leqslant S_1(c) \wedge S_2(c) \wedge S_3(c) \in \langle \bar{\mathscr{S}} \rangle.$$

Thus at (4.1), a can be written as a suprema of c's, each c dominated by a member of $\langle \bar{\mathscr{S}} \rangle$, and this member of $\langle \bar{\mathscr{S}} \rangle$ dominated by a $B(a \wedge v)$. ∎

COROLLARY 4.9. *Let $f\colon X_1 \to X_2$ be a function, $\phi^{-1}\colon L_2 \to L_1$ be a lattice morphism, $(X_1, L_1, \tau(\mathscr{U}_1))$ and $(X_2, L_2, \tau(\mathscr{U}_2))$ be quasi-uniform topological spaces, and $\mathscr{U}_2 = \langle\!\langle \mathscr{S} \rangle\!\rangle$. Then, (f, ϕ) is continuous iff F_{ϕ}^{-1} preserves the sub-basic nbhds from $\bar{\mathscr{S}}$.*

5. INDUCED QUASI-UNIFORMITIES AND UNIFORMITIES

DEFINITION 5.1. Let $\mathscr{U} \subset (L^X)^{(L^X)}$. Then put $\mathscr{U}^{-1} = \{U^{-1}\colon U \in \mathscr{U}\}$ and $\mathscr{U} \Delta \mathscr{U}^{-1} = \{U \Delta U^{-1}\colon U \in \mathscr{U}\}$.

PROPOSITION 5.1. *Let X and L be given and let $\mathscr{U}$ be a quasi-uniformity on X. The following statements hold:*

(1) *$\mathscr{U}^{-1}$ is a quasi-uniformity on X.*

(2) *$\mathscr{U}\, \Delta \mathscr{U}^{-1}$ is a basis for a uniformity on X.*

(3) *$\langle \mathscr{U}\, \Delta \mathscr{U}^{-1} \rangle$ is the smallest uniformity containing any subbasis of $\mathscr{U}$; it is also the smallest uniformity containing any subbasis of $\mathscr{U}^{-1}$.*

Proof. We check only condition (6) of Definition 2.3 for statement (2); the other details are clear or found in [22]. Let $U \Delta U^{-1}$ be given. There are $V, W \in \mathscr{U}$ such that $W \circ W \leqslant V$ and $V \circ V \leqslant U$. Now $W \leqslant W \circ W$, so $W \wedge W^{-1} \leqslant W \circ W$, which implies $W \Delta W^{-1} \leqslant W \circ W$. Then

$$(W \Delta W^{-1}) \circ (W \Delta W^{-1}) \leqslant (W \circ W) \circ (W \circ W) \leqslant V \circ V.$$

The fact $W^{-1} \leqslant (W \circ W)^{-1} = W^{-1} \circ W^{-1}$ may be used in a symmetric argument to establish

$$(W \Delta W^{-1}) \circ (W \Delta W^{-1}) \leqslant V^{-1} \circ V^{-1}.$$

It follows that

$$\begin{aligned}(W \Delta W^{-1}) \circ (W \Delta W^{-1}) &\leqslant (V \circ V)\, \Delta (V^{-1} \circ V^{-1}) \\ &= (V \circ V)\, \Delta (V \circ V)^{-1} \\ &\leqslant U \Delta U^{-1}. \quad \blacksquare\end{aligned}$$

PROPOSITION 5.2. *Let X_1, X_2, L_1, L_2, and (f, ϕ_1, ϕ_2) be given satisfying Definition* 3.1. II.(1), (2). *Then F^{-1} has the following properties:*

(1) *F^{-1} preserves suprema and hence inclusions.*

(2) *F^{-1} preserves infima.*

(3) *F^{-1} preserves the Δ operation as applied to the inclusion preserving members of $(L_2^{X_2})^{(L_2^{X_2})}$.*

Proof. Using the $\mathscr{C}$ notation of Lemma 2.3 we write

$$\begin{aligned}F^{-1}(B_1 \Delta B_2)(a) &= \Phi_f^{-1}((B_1 \Delta B_2)(\Phi(f(a)))) \\ &= \bigvee \{\Phi_f^{-1}((B_1 \wedge B_2)(h)) \colon h \in \mathscr{C}(\Phi(f(a)))\}\end{aligned}$$

and

$$\begin{aligned}&[F^{-1}(B_1)\, \Delta F^{-1}(B_2)](a) \\ &= \bigvee \{\Phi_f^{-1}(B_1(\Phi(f(g)))) \wedge \Phi_f^{-1}(B_2(\Phi(f(g)))) \colon g \in \mathscr{C}(a)\} \\ &= \bigvee \{\Phi_f^{-1}((B_1 \wedge B_2)(\Phi(f(g)))) \colon g \in \mathscr{C}(a)\}.\end{aligned}$$

Put $K = \Phi_f^{-1} \circ (B_1 \wedge B_2)$. To prove (3) we verify

$$\bigvee \{K(\Phi(f(g))) \colon g \in \mathscr{C}(a)\} = \bigvee \{K(h) \colon h \in C(\Phi(f(a)))\}.$$

For "$\geqslant$," note $\bigvee \Phi(f(\mathscr{C}(a))) = \Phi(f(\bigvee \mathscr{C}(a)) = \Phi(f(a))$. Therefore, $\forall h \in \mathscr{C}(\Phi(f(a)))$, $\exists \Phi(f(g)) \in \Phi(f(\mathscr{C}(a)))$, $h \leqslant \Phi(f(g))$, i.e., $\forall h \in \mathscr{C}(\Phi(f(a)))$, $\exists \Phi(f(g)) \in \Phi(f(\mathscr{C}(a)))$, $K(\Phi(f(g))) \geqslant K(h)$. For "$\leqslant$," let $h \in \mathscr{C}(\Phi(f(a)))$ and put $c_h = \bigvee \Phi^{-1}(h)$, $k_h = f^{-1}(c_h) \wedge a$. Then

$$\begin{aligned}\bigvee \{k_h \colon h \in \mathscr{C}(\Phi(f(a)))\} &= a \wedge \left(\bigvee \{f^{-1}(c_h) \colon h \in \mathscr{C}(\Phi(f(a)))\}\right) \\ &= a \wedge f^{-1}\left(\bigvee \{c_h \colon h \in \mathscr{C}(\Phi(f(a)))\}\right) \\ &= a.\end{aligned}$$

Thus $\forall g \in \mathscr{C}(a)$, $\exists h \in \mathscr{C}(\Phi(f(a)))$, $g \leqslant k_h$. So

$$\begin{aligned}K(\Phi(f(g))) &\leqslant K(\Phi(f(k_h))) \\ &\leqslant K(\Phi(f(f^{-1}(c_h))) \wedge \Phi(f(a))) \\ &\leqslant K(\Phi(c_h) \wedge \Phi(f(a))) \\ &= K(h \wedge \Phi(f(a))) \\ &= K(h)\end{aligned}$$

which verifies "$\leqslant$." ∎

PROPOSITION 5.3. *If in Proposition 5.2 we assume $\mathscr{S}$ is a subbasis [basis] for a quasi-uniformity on X_2, then $F^{-1}(\mathscr{S}) = \{F^{-1}(V) \colon V \in \mathscr{S}\}$ is a subbasis [basis] for a quasi-uniformity on X_1.*

Proof. Proposition 5.2 immediately gives conditions (2, 3, 5) of Definition 2.3. For condition (6), let $U = F^{-1}(V) \in F^{-1}(\mathscr{S})$ and let $B \in \mathscr{S}$, $B \circ B \leqslant V$. Since $F^{-1}(B \circ B) \leqslant F^{-1}(V)$, it suffices to show $F^{-1}(B) \circ F^{-1}(B) \leqslant F^{-1}(B \circ B)$. For this it is sufficient to show that on $L_2^{X_2}$, $\Phi \circ f_{\text{ind}} \circ \Phi_f^{-1} \leqslant i_{L_2^{X_2}}$. Let $b \in L_2^{X_2}$. It is routine to check that $\Phi_f^{-1}(b) \leqslant \phi_2^{-1} \circ b$. Thus

$$\begin{aligned}\Phi(f(\Phi_f^{-1}(b))) &\leqslant \Phi(\phi_2^{-1} \circ b) \\ &= \phi_1 \circ \phi_2^{-1} \circ b \\ &\leqslant b,\end{aligned}$$

where the last inequality follows from $\phi_1 \circ \phi_2^{-1} \leqslant i_{L_2}$ (Definition 3.1 II(2)). ∎

Remark 5.1. If we assume that f is a surjection and the $\phi_1 \circ \phi_2^{-1} = i_{L_2}$, then the above proof is strenghtened to show F^{-1} preserves $\circ$ on $L_2^{X_2}$ (cf. Proposition 5.2).

Remark 5.2. If f is a bijection, $L_1 = L = L_2$, and $\phi_1 = \phi_2 = i_L$, then F^{-1} preserves both inverses and supersets, but otherwise need not preserve either. Thus $F^{-1}(\mathscr{S})$ in Proposition 5.3 need not be a quasi-uniformity even if $\mathscr{S}$ is a uniformity.

Remark 5.3. Proposition 5.1(3) and Remark 5.2 imply that the obvious way of trying to map $\mathbb{QU}$ into $\mathbb{U}$ will not work (morphisms are not preserved). By the criteria of Remark 5.2 of [62], $\mathbb{QU}$ should be viewed as a strict generalization of $\mathbb{U}$.

Given the above results, the details of the proof of the following lemma are straightforward and omitted.

LEMMA 5.4. *Let $\{(X_\xi, L_\xi, \mathscr{U}_\xi)\}_\xi$ be a collection of objects of $\mathbb{QU}$, let $\mathscr{S}_\xi$ be a subbasis for $\mathscr{U}_\xi$ for each ξ, and let $(f_\xi, \phi_1^\xi, \phi_2^\xi)\colon (X, L) \to (X_\xi, L_\xi, \mathscr{U}_\xi)$ satisfy Definition* 3.1 II(1), (2) *for each ξ. Then the following statements hold*:

(1) *$\bigcup_\xi F_\xi^{-1}(\mathscr{S}_\xi)$ is a subbasis for a quasi-uniformity $\mathscr{Q}$ on X which is the smallest quasi-uniformity containing $\bigcup_\xi F_\xi^{-1}(\mathscr{S}_\xi)$ and making each $(f_\xi, \phi_1^\xi, \phi_2^\xi)$ quasi-uniformly continuous.*

(2) *$\mathscr{Q}$ induces (by Proposition* 5.1) *a uniformity $\mathscr{U}$ on X which is the smallest uniformity containing $\bigcup_\xi F_\xi^{-1}(\mathscr{S}_\xi)$, and if each $\mathscr{U}_\xi$ is a uniformity, making each $(f_\xi, \phi_1^\xi, \phi_2^\xi)$ uniformly continuous.*

COROLLARY 5.5. *Let X, $L_1 = L = L_2$, $\phi_1 = i_L = \phi_2$, $A \subset X$, and $i\colon A \hookrightarrow X$ the injection be given. If $\mathscr{Q}$ is a quasi-uniformity on X, then $I^{-1}(\mathscr{Q}) = \mathscr{Q}_A$, where I^{-1} is the auxiliary map of (i, ϕ_1, ϕ_2) and $\mathscr{Q}_A$ is defined as in Discussion* 3.1(1).

Proof. The details are straightforward given the computations $i(a) = a^e$ for $a \in L^A$, $b \circ i = b \mid A$ for $b \in L^X$, and $I^{-1}(U)(a) = U(a^e) \mid A$ for each $U \in \mathscr{Q}$. ∎

Remark 5.4. Although Corollary 5.5 implies that Lemma 5.1 subsumes Discussion 3.4(1), Lemma 5.7 below explicitly depends on Discussion 3.4(2). In contrast with Lemma 5.7, Discussion 3.4 avoids the use of nbhd systems in showing that the smallest topology making the injection $i\colon A \hookrightarrow X$ continuous is the topology induced by the smallest quasi-uniformity making i quasi-uniformly continuous.

Remark 5.5. Let X_1, L_1 be given, $(X_2, L_2, \mathscr{Q})$ be an object in $\mathbb{QU}$, and (f, ϕ_1, ϕ_2): $(X_1, L_1, \mathscr{Q}) \to (X_2, L_2, \mathscr{Q})$ satisfy Definition 3.1 II(1), (2). Then on X_1 the following uniformities are the same:

$$\langle F^{-1}(\mathscr{Q}\,\Delta\,\mathscr{Q}^{-1})\,\Delta F^{-1}(\mathscr{Q}\,\Delta\,\mathscr{Q}^{-1})^{-1}\rangle,$$

$$\langle\langle F^{-1}(\mathscr{Q})\,\Delta F^{-1}(\mathscr{Q}^{-1})\rangle\,\Delta\,\langle F^{-1}(\mathscr{Q})\,\Delta F^{-1}(\mathscr{Q}^{-1})\rangle^{-1}\rangle.$$

DEFINITION 5.2. Let $\{(X_\xi, L, \mathscr{Q}_\xi)\}_\xi$ be a collection of objects of $\mathbb{QU}(L, \phi_1, \phi_2)$, let X be the cartesian product $\times_\xi X_\xi$, let $\pi_\xi: X \to X_\xi$ be the projection, and let Π_ξ^{-1} be the auxiliary map of $(\pi_\xi, \phi_1, \phi_2)$. The (ϕ_1, ϕ_2)-*product quasi-uniformity* $\mathscr{Q}$ on X has subbasis $\bigcup_\xi \Pi_\xi^{-1}(\mathscr{Q}_\xi)$; the (ϕ_1, ϕ_2)-*product uniformity* $\mathscr{U}$ is induced by $\mathscr{Q}$. *If* $\phi_1 = \phi_2 = i_L$, $\mathscr{Q}$ is the *product quasi-uniformity* and $\mathscr{U}$ is the *product uniformity*.

PROPOSITION 5.6. *The following statements hold*:

(1) *The projections* $\{(\pi_\xi, \phi_1, \phi_2)\}_\xi$ *are quasi-uniformly [uniformly] continuous w.r.t. the* (ϕ_1, ϕ_2)-*product quasi-uniformity [uniformity]*.

(2) *The* (ϕ_1, ϕ_2)-*product quasi-uniformity induces the* ϕ_2-*product uniformity*.

Proof. The proof of (1) is straightforward; that of (2) is immediate given the following lemmas. ∎

LEMMA 5.7. *Let* (f, ϕ_1, ϕ_2): $(X_1, L_1, \mathscr{U}) \to (X_2, L_2, \mathscr{V})$ *in* $\mathbb{QU}$ *where* $\mathscr{U}$ *is the quasi-uniformity* $\langle F^{-1}(\mathscr{V})\rangle$, *and let* τ_w *be the smallest topology on* X *making* (f, ϕ_2): $(X_1, L_1, \tau_w) \to (X_2, L_2, \tau(\mathscr{V}))$ *continuous. Then* $\tau_w = \tau(\mathscr{U})$.

Proof. Immediately $\tau_w \subset \tau(\mathscr{U})$ by Proposition 3.2. For the reverse inclusion, let $a \in L_1^{X_1}$ and consider a basic nbhd of a in $\tau(\mathscr{U})$; we may assume by Proposition 5.3 and Proposition 4.8 that this basic nbhd is of the form

$$F^{-1}(V)(a) = \Phi_f^{-1}(V(\Phi(f(a)))),$$

where $V \in \mathscr{V}$. Note $V(\Phi(f(a)))$ is a basic nbhd of $\Phi(f(a))$ in $\tau(\mathscr{V})$. Since (f, ϕ_2) is continuous, $F^{-1}(V)(a)$ is a nbhd of $\Phi_f^{-1}(\Phi(f(a)))$ in τ_w (Proposition 4.7; $F_{\phi_2}^{-1} = \Phi_f^{-1}$). But the $\phi_2^{-1} \circ \phi_1 \geqslant i_{L_1}$ condition (Definition 3.1.II(2)) implies

$$\begin{aligned} a \leqslant f^{-1}(f(a)) &\leqslant (\phi_2^{-1} \circ \phi_1) \circ (f^{-1}(f(a))) \\ &= \phi_2^{-1} \circ \Phi(f(a)) \circ f \\ &= \Phi_f^{-1}(\Phi(f(a))). \end{aligned}$$

Thus $F^{-1}(V)(a)$ is a nbhd of a in τ_w. It follows that $\mathcal{N}(\tau(\mathcal{U})) \subset \mathcal{N}(\tau_w)$. By Corollary 4.6, $\tau(\mathcal{U}) \subset \tau_w$. ∎

LEMMA 5.8. *Let the collection of objects* $\{(X, L, \mathcal{Q}_\xi)\}_\xi$ *in* $\mathbb{QU}$ *be given, and let* $\mathcal{Q}$ *be the quasi-uniformity* $\langle\!\langle \bigcup_\xi \mathcal{Q}_\xi\} \rangle\!\rangle$. *Then* $\tau(\mathcal{Q}) = \bigvee_\xi \tau(\mathcal{Q}_\xi)$.

Proof. Because of Proposition 4.8, $\mathcal{N}(\tau(\mathcal{Q})) = \langle \bar{\mathcal{Q}} \rangle = \langle\!\langle \overline{\bigcup_\xi \mathcal{Q}_\xi} \rangle\!\rangle = \langle\!\langle \bigcup_\xi \bar{\mathcal{Q}}_\xi \rangle\!\rangle = \mathcal{N}(\bigvee_\xi \tau(\mathcal{Q}_\xi))$, where the last equality uses the complete distributivity of L. Apply Corollary 4.6. ∎

PROPOSITION 5.9. *Let the object* $(X, L, \mathcal{Q})$ *in* $\mathbb{QU}$ *be given, and let* $\mathcal{U}$ *be the uniformity* $\langle \mathcal{Q} \Delta \mathcal{Q}^{-1} \rangle$ *on* X. *Then* $\tau(\mathcal{U}) = \tau(\mathcal{Q}) \vee \tau(\mathcal{Q}^{-1})$.

Proof. Since $\langle \mathcal{Q} \Delta \mathcal{Q}^{-1} \rangle = \langle\!\langle \mathcal{Q} \cup \mathcal{Q}^{-1} \rangle\!\rangle$, Lemma 5.8 applies. ∎

6. APPLICATION TO THE FUZZY REAL LINES: ANSWER TO QUESTION A

DEFINITION 6.1. An object $(X, L, \mathcal{U})$ of $\mathbb{U}$ has a *pseudometric* or is *pseudometrizable* [24] if $\mathcal{U}$ has a basis $\{D_r: r \in \mathbb{R},\ t \geqslant 0\}$ of symmetric elements satisfying $D_r \circ D_s \leqslant D_{r+s}$ for each $r, s > 0$; in this case we speak of $(X, L, \tau(\mathcal{U}))$ as having a *pseudometric* or being *pseudometrizable*. A *pseudometric* [8] for such a pseudometrizable space is $d: L^X \times L^X \to [0, +\infty]$ defined by

$$d(a, b) = \bigwedge \{r: b \leqslant D_r(a)\}.$$

By (X, L, d) we mean the set X with the pseudometric d—we also call (X, L, d) a *pseudometric* space—and by $(X, L, \tau(d))$ we mean $(X, L, \tau(\mathcal{U}))$. The pseudometric spaces (X_1, L_1, d_1) and (X_2, L_2, d_2) are *isometric* if there is (f, ϕ) such that

(1) $f: X_1 \to X_2$ is a bijection,

(2) $\phi: L_1 \to L_2$ is an isomorphism,

(3) for each $(a, b) \in L_1^{X_1} \times L_2^{X_2}$, $d_1(a, b) = d_2(\Phi(f(a)), \Phi(f(b)))$, where Φ is defined as in Definition 3.1.II(3).

In this case, (f, ϕ) is an *isometry*.

PROPOSITION 6.1. *Let* $(X_1, L_1, \mathcal{U}_1)$, $(X_2, L_2, \mathcal{U}_2)$ *be pseudometrizable. Then*

$$(f, \phi, \phi): (X_1, L_1, \mathcal{U}_1) \to (X_2, L_2, \mathcal{U}_2)$$

is an isomorphism iff there are pseudometrics d_1, d_2 such that

$$(f, \phi): (X_1, L_1, d_1) \to (X_2, L_2, d_2)$$

is an isometry.

DEFINITION 6.2. Let L be given and define [22, 63] $B_\varepsilon: L^{\mathbb{R}(L)} \to L^{\mathbb{R}(L)}$ for real $\varepsilon > 0$ by

$$B_\varepsilon(a) = R_{t-\varepsilon}, \; t = \max\{s: a \leqslant (L_s)'\}.$$

Put $\mathscr{Q}(L) = \langle \{B_\varepsilon : \varepsilon > 0\} \rangle$ and $\mathscr{U}(L) = \langle \mathscr{Q}(L) \Delta \mathscr{Q}(L)^{-1} \rangle$ (by Proposition 5.1(2)).

Remark 6.1. From [22, 63] we have $\mathscr{Q}(L)$ [$\mathscr{Q}(L)^{-1}$] is a quasi-uniformity on $\mathbb{R}(L)$ generating the right-hand [left-hand] topology, $\mathscr{U}(L)$ is a uniformity on $\mathbb{R}(L)$ generating the canonical topology, and furthermore $\{B_\varepsilon \Delta B_\varepsilon^{-1} : \varepsilon > 0\}$ is a basis of $\mathscr{U}(L)$ satisfying Definition 6.1. So $\mathbb{R}(L)$ (as either a uniform or topological space) is pseudometrizable. We let $d(L)$ be the pseudometric induced by $\{B_\varepsilon \Delta B_\varepsilon^{-1} : \varepsilon > 0\}$. A property of $\{B_\varepsilon \Delta B_\varepsilon^{-1} : \varepsilon > 0\}$, crucial both for building $d(L)$ and for proving Theorem B (Sect. 7), is that for real $r, s > 0$, $B_r \circ B_s \leqslant B_{r+s}$, $B_r^{-1} \circ B_s^{-1} \leqslant B_{r+s}^{-1}$.

THEOREM A. *The following are equivalent*:

(1) L_1 *is isomorphic to* L_2.

(2) $(\mathbb{R}(L_1), L_1, \mathscr{U}(L_1))$ *is uniformly isomorphic to* $(\mathbb{R}(L_2), L_2, \mathscr{U}(L_2))$.

(3) $(\mathbb{R}(L_1), L_1, d(L_1))$ *is isometric to* $(\mathbb{R}(L_2), L_2, D(L_2))$.

Proof. (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1) are immediate. For (1) $\Rightarrow$ (2), let $\phi: L_1 \to L_2$ be an isomorphism, let $\lambda \in \mathbb{R}(L_1)$, and put $f(\lambda) = \phi \circ \lambda$. We claim (f, ϕ, ϕ) is a uniform isomorphism. Because of Proposition 5.2(1), (3), it suffices to show $F^{-1}(\mathscr{Q}(L_2)) = \mathscr{Q}(L_1)$, $F^{-1}(\mathscr{Q}(L_2)^{-1}) = \mathscr{Q}(L_1)^{-1}$, $F(\mathscr{Q}(L_1)) = \mathscr{Q}(L_2)$, $F(\mathscr{Q}(L_1)^{-1} = \mathscr{Q}(L_2)^{-1}$; we show only the first, breaking the proof into two parts. Let $\lambda \in \mathbb{R}(L_1)$ and $a \in L_1^{\mathbb{R}(L_1)}$:

I. $B_\varepsilon(\Phi(f(a)))(f(\lambda)) = R_{t-\varepsilon}(f(\lambda))$ in $\mathbb{R}(L_2)$ iff $B_\varepsilon(a)(\lambda) = R_{t-\varepsilon}(\lambda)$ in $\mathbb{R}(L_1)$. Since each of ϕ, ϕ^{-1} are order preserving bijections, we have

(i) $a(\lambda) \leqslant \lambda(s-) = L_s'(\lambda)$ iff

$$\begin{aligned}\phi(a(\lambda)) = \phi(a(f^{-1}(f(\lambda)))) = \phi(a(\phi^{-1} \circ f(\lambda))) &= (\phi \circ f(a))(f(\lambda)) \\ &\leqslant L_s'(f(\lambda)) = f(\lambda)(s-) = \phi(\lambda(s-))\end{aligned}$$

(ii) $t = \max\{s: \phi \circ f(a) \leqslant L'_s\}$ iff $t = \max\{s: a \leqslant L'_s\}$.

Claim I follows immediately.

II. $F^{-1}(B_\varepsilon)(a)(\lambda) = B_\varepsilon(a)(\lambda)$. Computing,

$$\begin{aligned}
F^{-1}(B_\varepsilon)(a)(\lambda) &= \phi^{-1}(B_\varepsilon(\phi \circ f(a)))(f(\lambda)) \\
&= (\phi^{-1} \circ B_\varepsilon(\phi \circ f(a)) \circ f)(\lambda) \\
&= \phi^{-1}(R_{t-\varepsilon}(f(\lambda))) \\
&= \phi^{-1}((\phi \circ \lambda)((t-\varepsilon)+)) \\
&= \lambda((t-\varepsilon)+) \\
&= R_{t-\varepsilon}(\lambda) \\
&= B_\varepsilon(a)(\lambda),
\end{aligned}$$

where the last equality follows from Claim I and the fact that $B_\varepsilon(\phi \circ f(a)) = R_{t-\varepsilon}$ (third equality). This concludes the proof of $(1) \Rightarrow (2)$.

For $(1) \Rightarrow (3)$, let ϕ, f be as in the proof of $(1) \Rightarrow (2)$. Then the proof that (f, ϕ) is an isometry follows using the details of $(1) \Rightarrow (2)$. ∎

COROLLARY A_1. *If a chain L_1 is isomorphic to L_2, then $\mathbb{R}(L_1)$ and $\mathbb{R}(L_2)$ are linearly uniformly isomorphic (and hence isometric and homeomorphic) complete fuzzy topological hyperfields.*

Proof. The mapping f used in the proof of Theorem A is the mapping J used in the proof of Theorem 5.2 of [64]; apply Theorem 5.2 of [64], Proposition 3.2, and Theorem A. ∎

COROLLARY A_2. *If a chain L_1 is isomorphic to a sublattice of L_2, then $\mathbb{R}(L_1)$ is linearly uniformly isomorphic to a complete fuzzy topological subhyperfield of $\mathbb{R}(L_2)$.*

7. APPLICATIONS TO THE FUZZY REAL LINES: ANSWER TO QUESTION B

In this section, L is always a complete chain.

LEMMA 7.1. *Let $\lambda, \mu \in \mathbb{R}(L)$. Then the following hold:*

(1) *For each $\alpha \in L$, there are $a(\lambda, \alpha)$, $b(\lambda, \alpha) \in [-\infty, +\infty]$ such that*

$$\begin{aligned}
\lambda(t-) \geqslant \alpha' \quad &\textit{iff } t \leqslant a(\lambda, \alpha) \\
\lambda(t+) \leqslant \alpha \quad &\textit{iff } t \geqslant b(\lambda, \alpha).
\end{aligned}$$

(2) *For each* $\alpha \in L$,

$$a(\lambda \oplus \mu, \alpha) = a(\lambda, \alpha) + a(\mu, \alpha)$$
$$b(\lambda \oplus \mu, \alpha) = b(\lambda, \alpha) + b(\mu, \alpha).$$

Proof. For (1), see Proposition 7.1 of [54]; for (2), see Theorem 4.1 of [60]. ∎

LEMMA 7.2. *For each* L_t, R_t *subbasic open sets in* $\mathbb{R}(L)$,

$$\oplus^{-1}(L_t) = \bigvee_{t_1 + t_2 = t} (\pi_1^{-1}(L_{t_1}) \wedge \pi_2^{-1}(L_{t_2}))$$

$$\oplus^{-1}(R_t) = \bigvee_{t_1 + t_2 = t} (\pi_1^{-1}(R_{t_1}) \wedge \pi_2^{-1}(R_{t_2})),$$

where $\pi_1, \pi_2 \colon \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ *are the projections.*

Proof. For $t \in (-\infty, +\infty)$, this is Theorem 2.2 of [42]. The cases $t = -\infty, +\infty$ are straightforward. ∎

THEOREM B. *Let* ϕ_1, ϕ_2 *satisfy Definition* 3.1.II(2) *for* $L_1 = L = L_2$, *let* $\mathbb{R}(L)$ *be equipped with the canonical uniformity* $\mathcal{U}(L)$ (*Definition* 6.2), *and let* $\mathbb{R}(L) \times \mathbb{R}(L)$ *be equipped with the* (ϕ_1, ϕ_2)*-product quasi-uniformity* $\mathcal{Q}$ *induced from* $\mathcal{U}(L)$ (*Definition* 5.2). *Then* $(\oplus, \phi_1, \phi_2)$ *is quasi-uniformly continuous.*

Proof. Let $\boxplus^{-1}$ denote the auxiliary map of $(\oplus, \phi_1, \phi_2)$. It suffices to show that for each (real) $\varepsilon > 0$, $\boxplus^{-1}(B_\varepsilon)$, $\boxplus^{-1}(B_\varepsilon^{-1}) \in \mathcal{Q}$. We only show $\boxplus^{-1}(B_\varepsilon) \in \mathcal{Q}$ (the other case is symmetric), and for this it suffices to show that

(i) $\boxplus^{-1}(B_\varepsilon)$ satisfies Definition 2.3(3)

(ii) $\boxplus^{-1}(B_\varepsilon)$ dominates some member of $\mathcal{Q}$ on $L^{\mathbb{R}(L) \times \mathbb{R}(L)}$.

Now (i) is immediate. For (ii), let $a \in L^{\mathbb{R}(L) \times \mathbb{R}(L)}$ and let $\pi_1, \pi_2 \colon \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ denote the projections. We proceed in two steps.

I. Fix $(\lambda_0, \mu_0) \in \mathbb{R}(L) \times \mathbb{R}(L)$ and $\alpha \in L$. Let $p_{(\lambda_0, \mu_0)}$ be the α-valued fuzzy point in $\mathbb{R}(L) \times \mathbb{R}(L)$ defined by

$$p_{(\lambda_0, \mu_0)}(\lambda, \mu) = \begin{cases} \alpha & (\lambda, \mu) = (\lambda_0, \mu_0) \\ 0 & \text{otherwise} \end{cases}$$

and let $t(\lambda_0, \mu_0)$, $w(\lambda_0, \mu_0)$, $z(\lambda_0, \mu_0)$ be defined by

$$t(\lambda_0, \mu_0) = \max\{s: \Phi(\oplus(p_{(\lambda_0, \mu_0)})) \leqslant L'_s\}$$
$$w(\lambda_0, \mu_0) = \max\{s: \Phi(\pi_1(p_{(\lambda_0, \mu_0)})) \leqslant L'_s\}$$
$$z(\lambda_0, \mu_0) = \max\{s: \Phi(\pi_2(p_{(\lambda_0, \mu_0)})) \leqslant L'_s\}.$$

Then we claim $t(\lambda_0, \mu_0) = w(\lambda_0, \mu_0) + z(\lambda_0, \mu_0)$. Let $\rho \in \mathbb{R}(L)$. We compute $\oplus(p_{(\lambda_0, \mu_0)})(\rho)$, $\pi_1(p_{(\lambda_0, \mu_0)})(\rho)$, and $\pi_2(p_{(\lambda_0, \mu_0)})(\rho)$. For the first computation, it follows that

$$\oplus(p_{(\lambda_0, \mu_0)})(\rho) = \bigvee \{p_{(\lambda_0, \mu_0)}(\lambda, \mu): \lambda \oplus \mu = \rho\}$$
$$= \begin{cases} \alpha, & \rho = \lambda_0 \oplus \mu_0 \\ 0, & \text{otherwise.} \end{cases}$$

Similar computations yield

$$\pi_1(p_{(\lambda_0, \mu_0)})(\rho) = \begin{cases} \alpha, & \lambda = \rho = \lambda_0,\ \mu = \mu_0 \\ 0, & \text{otherwise,} \end{cases}$$

$$\pi_2(p_{(\lambda_0, \mu_0)})(\rho) = \begin{cases} \alpha, & \mu = \rho = \mu_0,\ \lambda = \lambda_0 \\ 0, & \text{otherwise.} \end{cases}$$

Now let s be given such that $\Phi(\oplus(p_{(\lambda_0, \mu_0)})) \leqslant L'_s$. The only meaningful constraint on s is that $\Phi(\oplus(p_{(\lambda_0, \mu_0)}))(\lambda_0 \oplus \mu_0) \leqslant L'_s(\lambda_0 \oplus \mu_0)$. It follows $\phi_1(\alpha) \leqslant (\lambda_0 \oplus \mu_0)(s-)$. But by Lemma 7.1(1), this is true iff $s \leqslant a(\lambda_0 \oplus \mu_0, \phi_1(\alpha'))$. But then $t(\lambda_0, \mu_0) = a(\lambda_0 \oplus \mu_0, \phi_1(\alpha'))$. By Lemma 7.1(2),

$$t(\lambda_0, \mu_0) = a(\lambda_0, \phi_1(\alpha')) + a(\mu_0, \phi_1(\alpha')). \tag{7.1}$$

Using the computations of $\pi_1(p_{(\lambda_0, \mu_0)})$ and $\pi_2(p_{(\lambda_0, \mu_0)})$, and argumentation analogous to that preceding (7.1), we obtain

$$w(\lambda_0, \mu_0) = a(\lambda_0, \phi_1(\alpha')), \qquad z(\lambda_0, \mu_0) = a(\mu_0, \phi_1(\alpha')).$$

Immediately $t(\lambda_0, \mu_0) = w(\lambda_0, \mu_0) + z(\lambda_0, \mu_0)$.

II. For each $(\lambda_0, \mu_0) \in \mathbb{R}(L) \times \mathbb{R}(L)$, let $p_{(\lambda_0, \mu_0)}$ be the $a(\lambda_0, \mu_0)$-valued fuzzy point in $\mathbb{R}(L) \times \mathbb{R}(L)$ and put $\delta = \varepsilon/2$.
Note $a = \bigvee \{p_{(\lambda_0, \mu_0)}: (\lambda_0, \mu_0) \in \mathbb{R}(L) \times \mathbb{R}(L)\}$. So using Lemma 7.2,

$$\oplus^{-1}(B_\varepsilon(\Phi(\oplus(a))))$$

$$= \bigvee \{\oplus^{-1}(B_\varepsilon(\Phi(\oplus(p_{(\lambda_0, \mu_0)})))) : (\lambda_0, \mu_0) \in \mathbb{R}(L) \times \mathbb{R}(L)\}.$$

$$= \bigvee \{\oplus^{-1}(R_{t(\lambda_0, \mu_0) - \varepsilon})\}_{(\lambda_0, \mu_0)}$$

$$= \bigvee \left\{ \bigvee \{\pi_1^{-1}(R_{t_1}) \wedge \pi_2^{-1}(R_{t_2}) : t_1 + t_2 = t(\lambda_0, \mu_0) - \varepsilon\} \right\}_{(\lambda_0, \mu_0)}$$

$$\geqslant \bigvee \{\pi_1^{-1}(R_{w(\lambda_0, \mu_0) - \delta}) \wedge \pi_2^{-1}(R_{z(\lambda_0, \mu_0) - \delta})\}_{(\lambda_0, \mu_0)}$$

$$= \bigvee \{\pi_1^{-1}(B_\delta(\Phi(\pi_1(p_{(\lambda_0, \mu_0)})))) \wedge \pi_2^{-1}(B_\delta(\Phi(\pi_1(p_{(\lambda_0, \mu_0)}))))\}_{(\lambda_0, \mu_0)}$$

$$= \pi_1^{-1}(B_\delta(\Phi(\pi_1(a)))) \wedge \pi_2^{-1}(B_\delta(\Phi(\pi_2(a)))).$$

It follows that on $L^{\mathbb{R}(L) \times \mathbb{R}(L)}$,

$$\boxplus^{-1}(B_\varepsilon) \geqslant \Pi_1^{-1}(B_\delta) \wedge \Pi_2^{-1}(B_\delta)$$

and hence that

$$\boxplus^{-1}(B_\varepsilon) \geqslant \Pi_1^{-1}(B_\delta) \, \Delta \, \Pi_2^{-1}(B_\delta). \quad \blacksquare$$

Remark 7.1. Let $a \in L^{\mathbb{R}(L) \times \mathbb{R}(L)}$ and put

$$t = \max\{s : \Phi(\oplus(a)) \leqslant L'_s\}$$
$$w = \max\{s : \Phi(\pi_1(a)) \leqslant L'_s\}$$
$$z = \max\{s : \Phi(\pi_2(a)) \leqslant L'_s\}.$$

If $w + z = t$, then the proof that $\boxplus^{-1}(B_\varepsilon)(a) \geqslant (\Pi_1^{-1}(B_\delta) \, \Delta \, \Pi_2^{-1}(B_\delta))(a)$ is quite direct and does not need step I of the above proof. However, $w + z \geqslant t$ does not generally hold: if $L = \{0, 1\}$ and $a = \{(2, 1), (1, 2)\}$, then $w = 1 = z$, $t = 4$. On the other hand, $w + z \leqslant t$ is not difficult to verify. Note each element of $\mathbb{R}(L)$ may be written as some $\bar{\lambda} \oplus \bar{\mu}$ (since $\oplus$ is a surjection), so let $\bar{\lambda} \oplus \bar{\mu} \in \mathbb{R}(L)$. Then it follows

$$\Phi(\oplus(a))(\bar{\lambda} \oplus \bar{\mu}) = \bigvee_{\lambda \oplus \mu = \bar{\lambda} \oplus \bar{\mu}} (\phi_1(a(\lambda, \mu)))$$

$$\Phi(\pi_1(a))(\bar{\lambda}) = \bigvee_{\mu} \phi_1(a(\bar{\lambda}, \mu))$$

$$\Phi(\pi_2(a))(\bar{\mu}) = \bigvee_{\lambda} \phi_1(a(\lambda, \bar{\mu})).$$

Keeping in mind Theorem 2.1 of [42] and the definitions of w and z and letting $\lambda \oplus \mu = \bar{\lambda} \oplus \bar{\mu}$, where λ, μ are w.l.o.g. left-continuous representatives, we have

$$\begin{aligned}(\bar{\lambda} \oplus \bar{\mu})(w+z) &= \bigvee_{t_1 + t_2 = w + z} [\lambda(t_1) \wedge \mu(t_2)] \\ &\geqslant \lambda(w) \wedge \mu(z) \\ &\geqslant \phi_1(a(\lambda, \mu) \wedge \phi_1(a(\lambda, \mu)) \\ &= \phi_1(a(\lambda, \mu).\end{aligned}$$

Thus $\Phi(\oplus(a))(\bar{\lambda} \oplus \bar{\mu}) \leqslant (\bar{\lambda} \oplus \bar{\mu})(w+z) = L'_{w+z}(\bar{\lambda} \oplus \bar{\mu})$, i.e., $\Phi(\oplus(a)) \leqslant L'_{w+z}$. Hence $w + z \leqslant t$.

COROLLARY B. *The following hold*:

(1) $(\oplus, \phi_1, \phi_2)$ *is uniformly continuous with respect to the* (ϕ_1, ϕ_2)-*product uniformity* $\mathscr{U}$ *on* $\mathbb{R}(L) \times \mathbb{R}(L)$ *induced from* $\mathscr{Q}$.

(2) $(\oplus, \phi_2)$ *is continuous with respect to the* ϕ_2-*product topology induced on* $\mathbb{R}(L) \times \mathbb{R}(L)$ *from the canonical topology of* $\mathbb{R}(L)$.

(3) *For each L*, $\oplus$ *is continuous with respect to the canonical topology on* $\mathbb{R}(L)$ *and the canonical product topology on* $\mathbb{R}(L) \times \mathbb{R}(L)$.

(4) *In* (3), *replace "For each L" by "For L* = I*" and "product" by "star product."*

(5) *In* (2), (3), *and* (4), *replace "canonical" by "stratification of the canonical."*

Proof. For (1), clear; for (2), Theorem B and Propositions 3.2 and 5.6(2); for (3), (2) and the choice $\phi_1 = i_L = \phi_2$; for (4), (3) and Proposition 2.2; for (5)—(2), (3), (4), and Proposition 2.1. ∎

METACOROLLARY B. $\oplus$ *has a type of uniform continuity* (*namely quasi-uniform continuity*) *which implies its continuity both with respect to the canonical topologies and the stratification of the canonical topologies.*

Remark 7.2. The proof of Theorem B blends together both approaches to fuzzy addition [20, 42; 60].

8. APPLICATIONS TO THE FUZZY REAL LINES: ANSWER TO QUESTION C

The following definitions essentially parallel [25]. Recall a fuzzy point p_x^α with value $\alpha \in L$ and support $x \in X$ is defined by

$$p_x^\alpha(y) = \begin{cases} \alpha, & y = x \\ 0, & \text{otherwise.} \end{cases}$$

Also recall $L^c = \{\alpha \in L\colon \alpha \text{ is comparable to each element of } L\}$.

DEFINITION 8.1. Let $\mathscr{F} \subset L^X$. Then $\mathscr{F}$ is a *(proper) filter on* X (or a *prefilter* [35]) if $\mathscr{F} \neq \varnothing$, $0 \notin \mathscr{F}$, $\mathscr{F}$ is antiresidual, and $a, b \in \mathscr{F} \Rightarrow a \wedge b \in \mathscr{F}$.

DEFINITION 8.2. Let $(X, L, \mathscr{U}) \in |\mathbb{QU}|$, $\mathscr{U} = \langle \mathscr{B} \rangle$, and $\mathscr{F}$ be a filter on X.

(1) $\forall \alpha \in L$, $\forall U \in \mathscr{U}$, $\{U(p_x^\alpha): x \in X\}$ is an *α-uniform shading of X from $\mathscr{U}$* (cf. notion of α-shading in [11]).

(2) *$\mathscr{F}$ converges to p_x^α ($\mathscr{F} \to p_x^\alpha$) in $(X, L, \mathscr{U})$ if* $\forall U \in \mathscr{U}$, $U(p_x^\alpha) \in \mathscr{F}$. We also say *$\mathscr{F}$ α-converges.*

(3) $\mathscr{F}$ is *α-Cauchy* if $\mathscr{F}$ contains at least one element from each α-uniform shading of X from $\mathscr{U}$, or equivalently, $\forall U \in \mathscr{U}$, $\exists x \in X$, $U(p_x^\alpha) \in \mathscr{F}$.

(4) $\mathscr{F}$ is *weakly α-Cauchy w.r.t. $\mathscr{B}$ if* $\exists B \in \mathscr{B}$, $\exists x \in X$, $B(p_x^\alpha) \in \mathscr{F}$.

(5) $(X, L, \mathscr{U})$ is *α-complete* if each α-Cauchy filter α-converges; it is *strongly α-complete w.r.t. $\mathscr{B}$* if each filter α-converges which is weakly α-Cauchy w.r.t. $\mathscr{B}$; and it is *strongly α-complete* if there is a basis of $\mathscr{U}$ with respect to which it is strongly α-complete.

(6) If $\mathscr{U}$ is a uniformity, $\mathscr{B}$ satisfies Definition 6.1, and d is the induced pseudometric, then $(X, L, \mathscr{U})$ is *α-complete in d* if it is α-complete, and it is *strongly α-complete in d* if it is strong α-complete w.r.t. $\mathscr{B}$.

Remark 8.1. Because of Proposition 4.8, the above definitions may be restated using nbhds; e.g., $\mathscr{F} \to p_x^\alpha$ iff $\mathscr{F} \supset \mathscr{N}_{p_x^\alpha}(\tau(\mathscr{U}))$. Also note α-convergence $\Rightarrow$ α-Cauchy $\Rightarrow$ weakly α-Cauchy, and strongly α-complete in $d \Rightarrow$ strongly α-complete $\Rightarrow$ α-complete.

THEOREM C. *If $\alpha, \gamma \in L^c$ such that $\alpha > \gamma > \alpha'$, then the following hold:*

(1) $(\mathbb{R}(L), L, \mathscr{Q}(L))$ *is strongly α-complete.*

(2) $(\mathbb{R}(L), L, \mathscr{Q}(L)^{-1})$ *is strongly α-complete.*

(3) $(\mathbb{R}(L), L, \mathscr{U}(L))$ *is strongly α-complete in $d(L)$.*

Proof. We prove (3); (1) and (2) are similar. From Remark 6.1, $\{B_\varepsilon \varDelta B_\varepsilon^{-1}: \varepsilon > 0\}$ is a basis for $\mathscr{U}(L)$ satisfying Definition 6.1 and generating $d(L)$. Let $\mathscr{F}$ be a filter on $\mathbb{R}(L)$ which is weakly α-Cauchy w.r.t. this basis. Then there is $\delta > 0$ and $\lambda \in \mathbb{R}(L)$ such that $(B_\delta \varDelta B_\delta^{-1})(p_\lambda^\alpha) \in \mathscr{F}$. W.l.o.g., $\gamma \geqslant \gamma'$. Now let $\beta > \gamma$. Then $\beta > \beta'$ and hence $a(\lambda, \beta') \leqslant b(\lambda, \beta')$. Let $c < a(\lambda, \beta') - \delta$, define $\mu \in \mathbb{R}(L)$ by

$$\mu(s) = \begin{cases} 1, & s < c \\ \beta, & c < s < a(\lambda, \beta') - \delta \\ \gamma, & a(\lambda, \beta') - \delta < s < b(\lambda, \beta') + \delta \\ 0, & s > b(\lambda, \beta') + \delta \end{cases}$$

and put

$$t_\beta = \max\{s\colon p_\lambda^\beta \leqslant L_s'\}$$
$$w_\beta = \min\{s\colon p_\lambda^\beta \leqslant R_s'\}$$
$$t_\beta^* = \max\{s\colon p_\mu^\beta \leqslant L_s'\}$$
$$w_\beta^* = \min\{s\colon p_\mu^\beta \leqslant R_s'\}.$$

Then it is straightforward to verify as in the proof of Theorem B that

$$t_\beta^* = a(\mu, \beta') = a(\lambda, \beta') - \delta = t_\beta - \delta$$
$$w_\beta^* = b(\mu, \beta') = b(\lambda, \beta') + \delta = w_\beta + \delta.$$

Let $\varepsilon > 0$. Then

$$t_\beta^* - \varepsilon = t_\beta - \delta - \varepsilon < t_\beta - \delta$$
$$w_\beta^* + \varepsilon = w_\beta + \delta + \varepsilon > w_\beta + \delta$$
$$B_\varepsilon(p_\mu^\beta) \geqslant B_\delta(p_\lambda^\beta), \qquad B_\varepsilon^{-1}(p_\mu^\beta) \geqslant B_\delta^{-1}(p_\lambda^\beta)$$
$$(B_\varepsilon \wedge B_\varepsilon^{-1})(p_\mu^\beta) \geqslant (B_\delta \wedge B_\delta^{-1})(p_\lambda^\beta).$$

Thus

$$\begin{aligned}(B_\delta \mathbin{\Delta} B_\delta^{-1})(p_\lambda^\alpha) &= \bigvee \{(B_\delta \wedge B_\delta^{-1})(b)\colon b \in \mathscr{C}(p_\lambda^\alpha)\}\\ &= \bigvee \{(B_\delta \wedge B_\delta^{-1})(p_\lambda^\beta)\colon \beta \in \mathscr{C}(\alpha),\ \beta > \gamma > \gamma' > \beta'\}\\ &\leqslant \bigvee \{(B_\varepsilon \wedge B_\varepsilon^{-1})(p_\mu^\beta)\colon \beta \in \mathscr{C}(\alpha),\ \beta > \gamma > \gamma' > \beta'\}\\ &= \bigvee \{(B_\varepsilon \wedge B_\varepsilon^{-1})(b)\colon b \in \mathscr{C}(p_\mu^\alpha)\}\\ &= (B_\varepsilon \mathbin{\Delta} B_\varepsilon^{-1})(p_\mu^\alpha).\end{aligned}$$

It follows that $\mathscr{F} \to p_\mu^\alpha$. ∎

Remark 8.2. The hypotheses of Theorem C are commonly satisfied; e.g.,

(1) for $\alpha = 1$ for any L with either $|L| = 2$ or $|L^c| \geqslant 3$,

(2) for $\alpha > \frac{1}{2}$ for $L = I$.

Note (1) implies the completeness of $\mathbb{R}$ with the usual metric.

Remark 8.3. We conjecture [Theorem C(1) and (2)] $\not\Rightarrow$ Theorem C(3).

Remark 8.4. The above results should be compared with studies of the α-level properties of $\mathbb{R}(L)$ and $\mathbb{R}^c(L)$ as topological spaces; e.g., compare Theorem C with the results of [11, 54, 59, 61, 63] and especially compare Remark 8.2(2) with the results of [43].

9. Comments and Open Questions

In the ordinary or crisp case, there are two approaches to uniformities which coincide: entourages (as in [26]) and families of uniform coverings (as in [25]). In the fuzzy case, the category $\mathbb{L}$ of Lowen [38] generalizes for $L = I$ the entourage approach (a uniformity in [38] is a particular subset of $I^{X \times X}$), and the quasi-uniformities and uniformities of Hutton [22] generalize for each L the families of uniform coverings approach (each $U \in \mathscr{U}$ as a member of $(L^X)^{(L^X)}$ induces an α-uniform shading $\{U(p_x^\alpha)\colon x \in X\}$ (Definition 8.2)). Although the philosophical question of making the Hutton approach "categorically coherent" (tantamount to Question A) is solved in this paper by $\mathbb{QU}$ and $\mathbb{U}$, there remains

Question F. Is there a categorical framework which includes both $\mathbb{L}$ and $\mathbb{QU}$?

A weaker open question is

Question F′. Let $L = I$. Is there a common defintion satisfied by the objects of $\mathbb{L}$ and $\mathbb{U}(I, i_I, i_I)$?

A significant partial answer to Question F′ was obtained recently by Höhle [18]. Höhle shows that for $L = I$

(i) Hutton uniformities subject to two restrictions are fuzzy T_m-uniformities, where $T_m\colon I \times I \to I$ by $T_m(x, y) = \bigvee \{x + y - 1, 0\}$ is a continuous t-norm;

(ii) Lowen uniformities are Min-uniformities, where Min: $I \times I \to I$ by $\mathrm{Min}(x, y) = x \wedge y$ is a continuous t-norm;

and hence the Lowen uniformities and these restricted Hutton uniformities fit into the common framework of fuzzy T-uniformities, where $T\colon I \times I \to I$ is a continuous t-norm. Furthermore, Höhle shows that under additional restrictions, the fuzzy T-uniformities induce probabilistic metrics in the sense of [49, 65–67] on the underlying space. Finally, Höhle has pointed out in private communication that if Question F′ is stated for lattices having a multiplication and of countable type ($\forall A \subset L$, $\exists B \subset A$, $|B| \leqslant \aleph_0$ and $\bigvee B = \bigvee A$), then a partial answer is given by Definition 4.1 of [19] and Remark 2.2(a, b) of [16].

Concerning the generality of $\mathbb{QU}$ and $\mathbb{U}$, $\mathbb{U}$ is the smallest category in which Question A can be answered and it furnishes precisely that notion which for each real line generates both the canonical topology and the pseudometric generating the canonical topology. The generality of $\mathbb{QU}$ seems justified since it is the smallest coherent setting in which Question B can be answered and it includes uncountably many natural objects not in $\mathbb{U}$ (for each L with order reversing involution, $(\mathbb{R}(L), L, \mathscr{Q}(L))$,

$(\mathbb{R}(L), L, \mathscr{Q}(L)^{-1})$, $(I(L), L, \mathscr{Q}_{I(L)}(L))$, $(\mathbb{R}(L)\times\mathbb{R}(L), L, (\phi_1, \phi_2)$-product quasi-uniformity), etc; see [63]). The generality of $\mathbb{QU}$ and $\mathbb{U}$, since they (via J_0) furnish objects in $\mathbb{T}-\mathbb{T}_k$, further supports our positive answer to Question D (Sect. 1) and our negative answer to Question E (Sect. 1).

As for the morphisms of $\mathbb{QU}$ and $\mathbb{U}$, we note the importance of auxiliary maps in any setting for studying continuity. If f is a given map, not f itself, but some auxiliary map is used to determine the continuity of f. The auxiliary map is f^{-1} in ordinary continuity, the fuzzy continuity of [11, 13, 14, 21, 33, 54, 71], and the uniform continuity of [25], $(f\times f)^{-1}$ in the uniform continuity of [26] and in $\mathbb{L}$, F_ϕ^{-1} in $\mathbb{T}$, and F^{-1} in $\mathbb{QU}$ and $\mathbb{U}$. Note F^{-1} is a generalization of the f^{-1} of [25].

Concerning the relationship between $\mathbb{QU}$ and $\mathbb{T}$ (J_0 of Proposition 3.2), we should note from [22] that each object in $\mathbb{T}$ (for which the lattice is completely distributive) is quasi-uniformizable: for (X, L, τ), define $\mathscr{U}=\{U_u : u\in\tau\}$, where

$$U_u(a)=\begin{cases}1, & a\nleq u\\ u, & a\leqslant u.\end{cases}$$

This prompts

Question G. Does J_0 have a right or left adjoint?

Four more questions should be mentioned, if each L is a complete chain.

Question H. Clearly the fuzzy multiplication $\odot$ of [64] is generally not uniformly continuous (as the case $|L|=2$ shows). Now $\odot=\bigoplus_{i=1}^4 P_i$, where each $P_i\colon \mathbb{R}(L)\times\mathbb{R}(L)\to\mathbb{R}(L)$ is defined in [64]. Is each P_i (quasi-) uniform continuous on some nontrivial subset of $\mathbb{R}(L)\times\mathbb{R}(L)$? It is shown in [64] that $\odot$ is jointly continuous.

Question I. Is there a uniformity $\mathscr{U}^c$ on $\mathbb{R}^c(L)$ such that

(1) $\mathscr{U}^c(L)$ induces the stratified canonical topology;

(2) $\mathscr{U}^c(L)$ induces a pseudometric inducing the stratified canonical topology;

(3) $\mathscr{U}^c(L)$ induces a product quasi-uniformity on $\mathbb{R}^c(L)\times\mathbb{R}^c(L)$ which both makes $\oplus$ quasi-uniformly continuous and induces the stratified canonical product topology (so that Question B stated in $\mathbb{T}_k$ for $\mathbb{R}^c(L)$ can be answered in $\mathbb{T}_k$)?

Höhle conjectures in private communication that Question I has an affirmative answer if L is a complete Boolean algebra; cf. Proposition 5.2 of [16] and Theorem 6.3 and Remark 6.6 of [19].

Question J. Let $L=I$. Does $\oplus$ have any sort of uniform continuity with respect to a "star product" (quasi-) uniformity which would imply its continuity with respect to the star product topology (Definition 2.2(2))?

Question K. Let $L = I$ and let $\mathcal{U}$ be any T-uniformity [18] on $\mathbb{R}(I)$, where $T: I \times I \to I$ is a continuous t-norm. Is there an induced product T-uniformity on $\mathbb{R}(I) \times \mathbb{R}(I)$ making $\oplus$ uniformly continuous? Compare the results of [18] with Theorem 6.10 of [38] and Corollary B(1) above.

Note added in proof. We conjecture that $\mathbb{QU}(L, \phi_1, \phi_2)$ $[\mathbb{U}(L, \phi_1, \phi_2)]$ can be meaningfully restricted to a category which has good properties (completeness, cocompleteness,...) and are embeddable into $\mathbb{QU}$ $[\mathbb{U}]$ as a subcategory of $\mathbb{QU}(L, i_L, i_L)$ $[\mathbb{U}(L, i_L, i_L)]$; the same has already been done for $\mathbb{T}(L, \phi)$ w.r.t. $\mathbb{T}$ and $\mathbb{T}(L, i_L)$—see the author's Errata, *Fuzzy Sets and Systems* **20** (1986), 107–108.

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